ON ENTIRE FUNCTIONS DEFINED BY A
DIRICHLET SERIES

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1. The first part of the Theorem 2 of Rahman [1] can be improved and we may state the following result:

**Theorem A.** If \( h = \infty \), then the type \( T_S \) of \( f(s) \) in each horizontal strip \( S(\pi a) \), with \( a > 0 \), satisfies \( T_S = T \).

**Proof.** If \( a > a' > 0 \), evidently

\[ M_S(\sigma) \geq M_S'(\sigma) \]

where \( S = S(\pi a) \) and \( S' = S(\pi a') \); and therefore

\[ T_S \geq T_{S'} \]

From Theorem 2 of Rahman it follows

\[ T_S \geq T_{S'} \geq e^{-\pi a'T} \]

and hence, if \( a' \to 0 \),

\[ T_S \geq \lim T_{S'} \geq T. \]

But obviously \( T_S \leq T \), and therefore \( T_S = T \).

2. The Theorem A is included in the following theorem.

**Theorem B.** If we suppose \( h > 0 \), in each horizontal strip \( S(\pi a) \), with \( a > D \), the type \( T_S \) satisfies

\[ T \geq T_S \geq e^{-\beta a'T} \]

where \( \beta = \pi D + D(7 - 3 \cdot \log(hD)) \).

If \( h = \infty \), we have \( D = 0 \) and therefore \( \beta = 0 \), and since from Theorem B it follows that \( T_S = T \), Theorem B includes Theorem A, as we have said before.

**Proof of Theorem B.** We use the notations of Mandelbrojt [2]. According to a result of Mandelbrojt [2, Theorem a] for any \( s_0 = \sigma_0 + i\xi_0 \), inside the circle

\[ |s - s_0| \leq \pi D + \epsilon, \quad (1) \]

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\(^1\) Numbers in brackets refer to the bibliography at the end of the paper.

\(^2\) For notations see [1].
where \( \epsilon \) is a given positive quantity, arbitrarily small, but fixed, there will exist a point \( s' \) at which\(^8\)

\[
\log |f(s')| > \log |a_n| - \lambda_n \sigma_0 - \log (\lambda_n \Lambda_0^*) - c\epsilon,
\]

where \( c\epsilon \) is a constant which depends on \( \epsilon \); and this inequality will hold for every value of \( n \).

On the other hand, according to a result contained in the same paper of Mandelbrojt [2, p. 355] for values of \( n \) sufficiently large

\[
(3) \quad - \log (\lambda_n \Lambda_0^*) - c\epsilon > - \lambda_n (D(7 - 3 \log (hD)) + \epsilon).
\]

Then evidently, if \( \sigma \) is smaller than a certain negative quantity, the value of \( n \) which maximizes the expression:

\[
\log |a_n| - \lambda_n \sigma
\]

will be one of those which satisfy (3). As a consequence of (2), we conclude that, if \( \sigma_0 \) is smaller than a negative quantity, in the circle (1) there will exist a point \( s' \) at which the following inequality will hold:

\[
\log |f(s')| > \log \mu (\sigma_0 + d + \epsilon),
\]

where \( d = D(7 - 3 \log (hD)) \). Moreover, according to Sugimura [3, Theorem 5], as \( D \) is finite,

\[
\log \mu (\sigma) = (1 - o(1)) \log M(\sigma)
\]

and consequently we shall have

\[
\log |f(s')| > (1 - o(1)) \log M(\sigma_0 + d + \epsilon).
\]

Therefore if we write \( s' = \sigma' + it' \), we shall have:

\[
\log M_\beta (\sigma') > (1 - o(1)) \log M(\sigma_0 + d + \epsilon).
\]

And, since \( \sigma' \geq \sigma_0 - \pi D - \epsilon \), and \( \epsilon \) is arbitrary,

\[
T_\beta \geq e^{-\beta \pi} T.
\]

On the other hand, the inequality

\[
T \geq T_\beta
\]

is evident.

3. On representing by \( \Delta \) the maximum density of \( \{\lambda_n\} \), introduced by Pólya [4], we can state the following theorem.

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\(^8\) When the Dirichlet series contains a constant term the definition of the \( \Delta(\tau) \) and, therefore, of the \( \Lambda_0^* \) given by Mandelbrojt must vary slightly.
Theorem C. If we suppose \( h > 0 \), in each horizontal strip \( S(\pi a) \), with \( a > \Delta \), the type \( T_S \) satisfies \( T_S = T \).

Since \( \Delta \) can be \( \geq \Delta \), this theorem does not contain Theorem B.

Theorem C is a corollary of my generalization of a result due to Pólya [5, Lemmas 2, 3].

4. The proofs of Theorem 1 and of the second part of Theorem 2 of Rahman are not complete; since \( \sigma_j^* \) can be a discontinuous function of \( \sigma_j \), we can neither affirm that

\[
\lim_{\sigma_j \to -\infty} \inf \log \log M_S(\sigma_j^*) = \lambda_S
\]

nor that

\[
\lim_{\sigma_j \to -\infty} \inf \log \frac{M_S(\sigma_j^*)}{e^{-\rho \sigma_j}} = \tau_S
\]

and therefore we can neither affirm that \( \lambda_S \geq \lambda \) nor that \( \tau_S \geq e^{-\rho \sigma_T} \).

Remark. By the same method similar results are proved for the series of Dirichlet and for the integral of Laplace.

Bibliography

5. F. Sunyer i Balaguer, Sobre la distribución de los valores de una función entera representada por una serie de Dirichlet lagunar, Rev. Acad. Ci. Zaragoza (2) vol. 5 (1950) pp. 25–73.

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* The results however might be exact; particularly if \( \rho < \infty \) I think it likely they are exact, but the proofs of these results seem to be rather difficult.