ON ENTIRE FUNCTIONS DEFINED BY A DIRICHLET SERIES

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1. The first part of the Theorem 2 of Rahman [1][1] can be improved and we may state the following result:[2]

THEOREM A. *If* $h=\infty$, *then the type* T_S *of* $f(s)$ *in each horizontal strip* $S(\pi a)$, *with* $a>0$, *satisfies* $T_S=T$.

PROOF. If $a>a'>0$, evidently

$$M_S(\sigma) \geqq M_{S'}(\sigma)$$

where $S=S(\pi a)$ and $S'=S(\pi a')$; and therefore

$$T_S \geqq T_{S'}.$$

From Theorem 2 of Rahman it follows

$$T_S \geqq T_{S'} \geqq e^{-\pi\rho a'}T$$

and hence, if $a'\to 0$,

$$T_S \geqq \lim T_{S'} \geqq T.$$

But obviously $T_S \leqq T$, and therefore $T_S=T$.

2. The Theorem A is included in the following theorem.

THEOREM B. *If we suppose* $h>0$, *in each horizontal strip* $S(\pi a)$, *with* $a>D$, *the type* T_S *satisfies*

$$T \geqq T_S \geqq e^{-\beta\rho}T$$

where $\beta=\pi D+D(7-3\cdot\log(hD))$.

If $h=\infty$, we have $D=0$ and therefore $\beta=0$, and since from Theorem B it follows that $T_S=T$, Theorem B includes Theorem A, as we have said before.

PROOF OF THEOREM B. We use the notations of Mandelbrojt [2]. According to a result of Mandelbrojt [2, Theorem a] for any $s_0=\sigma_0+it_0$, inside the circle

$$|s-s_0| \leqq \pi D+\epsilon, \tag{1}$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] For notations see [1].

where ϵ is a given positive quantity, arbitrarily small, but fixed, there will exist a point s' at which[3]

$$\log |f(s')| > \log |a_n| - \lambda_n\sigma_0 - \log (\lambda_n \Lambda_n^*) - c_\epsilon, \tag{2}$$

where c_ϵ is a constant which depends on ϵ; and this inequality will hold for every value of n.

On the other hand, according to a result contained in the same paper of Mandelbrojt [2, p. 355] for values of n sufficiently large

$$-\log (\lambda_n\Lambda_n^*) - c_\epsilon > -\lambda_n(D(7 - 3 \log (hD)) + \epsilon). \tag{3}$$

Then evidently, if σ is smaller than a certain negative quantity, the value of n which maximizes the expression:

$$\log |a_n| - \lambda_n\sigma$$

will be one of those which satisfy (3). As a consequence of (2), we conclude that, if σ_0 is smaller than a negative quantity, in the circle (1) there will exist a point s' at which the following inequality will hold:

$$\log |f(s')| > \log \mu(\sigma_0 + d + \epsilon),$$

where $d = D(7-3\log(hD))$. Moreover, according to Sugimura [3, Theorem 5], as D is finite,

$$\log \mu(\sigma) = (1 - o(1)) \log M(\sigma)$$

and consequently we shall have

$$\log |f(s')| > (1 - o(1)) \log M(\sigma_0 + d + \epsilon).$$

Therefore if we write $s' = \sigma' + it'$, we shall have:

$$\log M_S(\sigma') > (1 - o(1)) \log M(\sigma_0 + d + \epsilon).$$

And, since $\sigma' \geqq \sigma_0 - \pi D - \epsilon$, and ϵ is arbitrary,

$$T_S \geqq e^{-\beta\rho} T.$$

On the other hand, the inequality

$$T \geqq T_S$$

is evident.

3. On representing by Δ the maximum density of $\{\lambda_n\}$, introduced by Pólya [4], we can state the following theorem.

[3] When the Dirichlet series contains a constant term the definition of the $\Lambda(r)$ and, therefore, of the Λ_n^* given by Mandelbrojt must vary slightly.

THEOREM C. *If we suppose $h>0$, in each horizontal strip $S(\pi a)$, with $a>\Delta$, the type T_S satisfies $T_S=T$.*

Since Δ can be $>D$, this theorem does not contain Theorem B.

Theorem C is a corollary of my generalization of a result due to Pólya [**5**, Lemmas 2, 3].

4. The proofs of Theorem 1 and of the second part of Theorem 2 of Rahman are not complete;[4] since σ_j^* can be a discontinuous function of σ_j, we can neither affirm that

$$\liminf_{\sigma_j\to-\infty}\frac{\log\log M_S(\sigma_j^*)}{-\sigma_j^*}=\lambda_S$$

nor that

$$\liminf_{\sigma_j\to-\infty}\frac{\log M_S(\sigma_j^*)}{e^{-\rho\sigma_j^*}}=\tau_S$$

and therefore we can neither affirm that $\lambda_S\geqq\lambda$ nor that $\tau_S\geqq e^{-\pi\rho a}\tau$.

REMARK. By the same method similar results are proved for the series of Dirichlet and for the integral of Laplace.

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[4] The results however might be exact; particularly if $\rho<\infty$ I think it likely they are exact, but the proofs of these results seem to be rather difficult.