ON ENTIRE FUNCTIONS DEFINED BY A
DIRICHLET SERIES

F. SUNYER I BALAGUER

1. The first part of the Theorem 2 of Rahman [1] can be improved and we may state the following result:

**Theorem A.** If \( h = \infty \), then the type \( T_S \) of \( f(s) \) in each horizontal strip \( S(\pi a) \), with \( a > 0 \), satisfies \( T_S = T \).

**Proof.** If \( a > a' > 0 \), evidently

\[
M_S(\sigma) \geq M_{S'}(\sigma)
\]

where \( S = S(\pi a) \) and \( S' = S(\pi a') \); and therefore

\[
T_S \geq T_{S'}.
\]

From Theorem 2 of Rahman it follows

\[
T_S \geq T_{S'} \geq e^{-\rho a'} T
\]

and hence, if \( a' \to 0 \),

\[
T_S \geq \lim T_{S'} \geq T.
\]

But obviously \( T_S \leq T \), and therefore \( T_S = T \).

2. The Theorem A is included in the following theorem.

**Theorem B.** If we suppose \( h > 0 \), in each horizontal strip \( S(\pi a) \), with \( a > D \), the type \( T_S \) satisfies

\[
T \geq T_S \geq e^{-\beta \rho T}
\]

where \( \beta = \pi D + D(7 - 3 \cdot \log(hD)) \).

If \( h = \infty \), we have \( D = 0 \) and therefore \( \beta = 0 \), and since from Theorem B it follows that \( T_S = T \), Theorem B includes Theorem A, as we have said before.

**Proof of Theorem B.** We use the notations of Mandelbrojt [2]. According to a result of Mandelbrojt [2, Theorem a] for any \( s_0 = \sigma_0 + i \delta_0 \), inside the circle

\[
|s - s_0| \leq \pi D + \epsilon,
\]

Received by the editors July 10, 1959 and, in revised form, October 30, 1959.

Numbers in brackets refer to the bibliography at the end of the paper.

For notations see [1].
where $\epsilon$ is a given positive quantity, arbitrarily small, but fixed, there will exist a point $s'$ at which²

\[(2) \quad \log |f(s')| > \log |a_n| - \lambda_n\sigma_0 - \log (\lambda_n \Lambda_n^*) - c_\epsilon,\]

where $c_\epsilon$ is a constant which depends on $\epsilon$; and this inequality will hold for every value of $n$.

On the other hand, according to a result contained in the same paper of Mandelbrojt [2, p. 355] for values of $n$ sufficiently large

\[(3) \quad -\log (\lambda_n\Lambda_n^*) - c_\epsilon > -\lambda_n(D(7 - 3 \log (hD)) + \epsilon).\]

Then evidently, if $\sigma$ is smaller than a certain negative quantity, the value of $n$ which maximizes the expression:

\[\log |a_n| - \lambda_n\sigma\]

will be one of those which satisfy (3). As a consequence of (2), we conclude that, if $\sigma_0$ is smaller than a negative quantity, in the circle (1) there will exist a point $s'$ at which the following inequality will hold:

\[\log |f(s')| > \log \mu(\sigma_0 + d + \epsilon),\]

where $d = D(7 - 3 \log (hD))$. Moreover, according to Sugimura [3, Theorem 5], as $D$ is finite,

\[\log \mu(\sigma) = (1 - o(1)) \log M(\sigma)\]

and consequently we shall have

\[\log |f(s')| > (1 - o(1)) \log M(\sigma_0 + d + \epsilon).\]

Therefore if we write $s' = \sigma' + it'$, we shall have:

\[\log M_{s'}(\sigma') > (1 - o(1)) \log M(\sigma_0 + d + \epsilon).\]

And, since $\sigma' \geq \sigma_0 - \pi D - \epsilon$, and $\epsilon$ is arbitrary,

\[T_S \geq e^{-\beta_0} T.\]

On the other hand, the inequality

\[T \geq T_S\]

is evident.

3. On representing by $\Delta$ the maximum density of $\{\lambda_n\}$, introduced by Pólya [4], we can state the following theorem.

² When the Dirichlet series contains a constant term the definition of the $\Lambda(r)$ and, therefore, of the $\Lambda_n^*$ given by Mandelbrojt must vary slightly.
Theorem C. If we suppose \( h > 0 \), in each horizontal strip \( S(\pi a) \), with \( a > \Delta \), the type \( T_S \) satisfies \( T_S = T \).

Since \( \Delta \) can be \( > D \), this theorem does not contain Theorem B.

Theorem C is a corollary of my generalization of a result due to Pólya [5, Lemmas 2, 3].

4. The proofs of Theorem 1 and of the second part of Theorem 2 of Rahman are not complete; since \( \sigma_j^* \) can be a discontinuous function of \( \sigma_j \), we can neither affirm that

\[
\liminf_{\sigma_j \to \infty} \frac{\log \log M_S(\sigma_j^*)}{-\sigma_j^*} = \lambda_S
\]

nor that

\[
\liminf_{\sigma_j \to \infty} \frac{\log M_S(\sigma_j^*)}{e^{-\rho} \sigma_j^*} = \tau_S
\]

and therefore we can neither affirm that \( \lambda_S \geq \lambda \) nor that \( \tau_S \leq e^{-\rho} \). 

Remark. By the same method similar results are proved for the series of Dirichlet and for the integral of Laplace.

Bibliography


5. F. Sunyer i Balaguer, Sobre la distribución de los valores de una función entera representada por una serie de Dirichlet lagunar, Rev. Acad. Ci. Zaragoza (2) vol. 5 (1950) pp. 25–73.

University of Barcelona, Barcelona, Spain

4 The results however might be exact; particularly if \( \rho < \infty \) I think it likely they are exact, but the proofs of these results seem to be rather difficult.