VARIATIONAL COMPLETENESS FOR COMPACT
SYMMETRIC SPACES

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We prove the following theorem: Let $K$ be a connected symmetric
subgroup of the group of isometries of a compact connected globally
Riemannian symmetric space $M$. Then, $K$ acts in a variationally com-
plete manner on $M$. (We refer to the work of R. Bott and H. Samelson
[1; 2] for a definition of this concept and applications to topology.)

Let $G$ be a compact, connected Lie group, $L$ a connected symmetric
subgroup of $G$. Bott and Samelson have proved the following: (1) $L$
acts in a variationally complete manner on $G/L$, the right coset space,
(2) $L$ acts via the linear isotropy representation in a variationally
complete way on the tangent space at a point of $G/L$.

Our theorem then generalizes (1), but does not imply (2). The
method is essentially the same as Bott's [1], but uses a Lie algebra
approach in a stronger way. The proof then provides an exposition
of Bott's very important result from a Lie-algebra point of view.
Notice also that it suggests the general program of studying the
Jacobi-fields on a symmetric space as a problem of Lie algebra theory,
even in the nonvariational complete case.

1. We briefly recall Bott's notations: If $p \in M$, $M_p$ denotes the
tangent space to $M$ at $p$. Let $O_p(K)$ be the orbit of $K$ at $p$, let
$O(K)_p$ be its tangent space at $p$, and let $g: R \rightarrow M$ be a geodesic of $M$ begin-
ning at $p$, perpendicular to $O_p(K)$. Consider the vector space $J_g$
of Jacobi vector fields along $g$. A Jacobi field is a map $t \mapsto Y_t \in M_g(t)$ for
$t \in R$ that is a solution of the Jacobi Equation:

\[(1.1) \quad Y'' + R(X_t, Y_t)(X_t) = 0,\]

where $t \mapsto X_t$ is the tangent vector field to the geodesic $g$, $R(X_t, Y_t)$ is
the linear map $M_g(t) \rightarrow M_g(t)$ defined by the curvature form evaluated
at $(X_t, Y_t)$ and $t \mapsto Y_t''$ is the second covariant derivative of the field
$Y_t$ along $g$.

$J^K_g$ denotes the focal subspace of $J_g$ relative to $O(K)$, i.e. $J^K_g$
consists of those fields $Y_t$ satisfying the initial conditions

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\[ Y_0, Y_0' + T_g(Y_0) \in O(K)_p, \text{ where } T_g \text{ is a certain linear transformation } O(K)_p \to O(K)_p. \]

Hence, \( \dim J_g^K = \dim M. \)

Let \( K \) denote the Lie algebra of \( K \). There is a linear mapping \( \pi: K \to J_g^K \) such that for \( k \in K \), \( \pi(k) \) is obtained by restricting the vector field determined by \( A \) on \( J_g^K \) to \( g \).

By definition, to prove variational completeness we must show that every element of \( J_g^K \) that vanishes at some point of \( g \) must lie in \( \pi(K) \).

2. \( M \) is a coset space \( G/L \), where \( L \) is a compact, symmetric subgroup of a compact, connected Lie group \( G \). It evidently suffices to prove the theorem in the case where \( p \) is the identity coset.

We have a natural reduction \( G = L \oplus M \), with \( [M, M] \subset L \) and \( [L, M] \subset M \). (For the ideas and results of the differential geometry of symmetric spaces, see [3].) \( M_p \) can be identified with \( M \), and each \( M_{\theta(t)} \), for \( t \in R \), can be identified with \( M_p \) by parallel transport along \( g \). There is then a correspondence between vector fields along \( g \) and curves \( t \to Y_t \) in \( M \). The metric on \( M \) can be considered as induced by a positive definite quadratic form on \( G \) invariant under \( \text{Ad} \ G \).

Let \( P \) denote the projection of \( G \) on \( M \). Then \( N = P(K) \) is identified with \( O(K)_p \). Let \( X \in M \) correspond to \( X_0 \). Then \( X \in N^X \). One sees that a curve \( t \to Y_t \) in \( M \) corresponds to a vector field in \( J_g^K \) if and only if

\[
Y_0 \in N, \\
Y_0' + T_g(Y_0) \in N^\perp, \\
Y_0'' = (\text{Ad} X)^2 Y_t,
\]

where \( t \to Y_t' \) is now the derivative in the ordinary sense. (One uses the explicit formula \( \text{Ad} [x, y] = -R(x, y) \) for the curvature in symmetric spaces and the fact that curvature is invariant under parallel translation in identifying \( R(X_t, Y_t)X_t \) with \( (\text{Ad} X)^2 Y_t, [3]. \))

If \( k \in K \), \( \pi(k) \) corresponds to the curve \( t \to \pi(k)_t = \text{P}((\text{Ad} \text{Exp} tX)(k)) \), hence \( \pi(k)_0 = P(k), \pi(k)'_0 = P([X, k]) \).

The question of variational completeness can now be treated in this Lie algebra setting as a property of solutions of vector-valued ordinary linear differential equations (2.2) with constant coefficients. In particular, a reduction of \( M \) into subspaces invariant under
(Ad X)² leads to a decomposition of solutions of (2.2) into solutions taking values in the invariant subspaces.

3. The proof.
(a) kernel \( \pi = K \cap L \cap \text{kernel Ad } X. \)
(b) \( \dim \pi(K) = \dim K - \dim K \cap L \cap \text{kernel Ad } X = \dim P(K) + \dim K \cap L \cap \text{kernel Ad } X = \dim P(K) + \dim \text{Ad } X(K \cap L). \)
(c) \( [K_L, K_L] \subset K, [K, K_L] \subset K_L, \) since \( K, \) is a symmetric subalgebra of \( G. \) (The perpendicular operation \( \perp \) is always with respect to the given metric on \( G. \) \( L_L = M. \) \( \text{Ad } X)^2(P(K)) \subset P(K), \) since \( X \in K_L \cap L. \)
(d) \( \text{Ad } X(K \cap L) \subset K_L \cap L \) hence \( \text{Ad } X(K \cap L) \subset P(K) \cap M. \) \( \text{Ad } X)^2(\text{Ad } X(K \cap L)) \subset (\text{Ad } X)^2(K_L \cap L) \subset \text{Ad } X(K \cap L). \)
(e) Define \( Q = M \cap P(K) \cap \text{Ad } X(K \cap L) = (P(K) + \text{Ad } X(K \cap L)) \cap M. \) Then, \( (\text{Ad } X)^2(Q) \subset Q, Q \subset K_L \cap L. \)
(f) \( \text{Ad } X(Q) = 0, \) for, \( \text{Ad } X(Q) \subset K \cap L, \) hence \( \text{Ad } X^2(Q) \subset K \cap L. \)

\( \text{Dim } M = \dim M = \dim J^K_0 = \dim (\pi(K) + Q). \)

(h) If \( k \subset K, \) then \( \pi(k) \subset Q \) for all \( t \geq 0. \)

Proof. \( \pi(k)_t = P(\sinh(\text{Ad } X)(k)) + P(\cosh(\text{Ad } X)(k)). \) Now, \( P(\cosh(\text{Ad } X)(k)) \subset P(K) \subset Q. \) Then, \( \sinh(\text{Ad } X)(k) \subset Q \) because of (e) and (f), and the fact that \( (\text{Ad } X)^2 \) is a symmetric transformation, with respect to the positive-definite quadratic form on \( G, \) that commutes with \( P. \)

Now, define a map \( \psi : Q \rightarrow J^K_0 \) as follows: For \( q \in Q, \psi(q), \) is the curve in \( M \) satisfying (2.2) and
\[
\psi(q_0) = 0,
\psi(q') = q.
\]

Because of (e) and (f), \( \psi(q)_t = tq, \) and hence \( \psi \) is one-to-one. Then \( J^K_0 = \psi(Q) + \pi(K), \psi(Q) \cap \pi(K) = 0 \) and the theorem follows. For if \( t \rightarrow Y_t \) is an element of \( J^K_0, Y_t = \pi(k)_t + t q \) for \( k \subset K, q \subset Q, \) and \( Y_t = 0 \) for some \( t > 0, \) then \( q = 0 \) by (h).

Bibliography