VARIATIONAL COMPLETENESS FOR COMPACT SYMMETRIC SPACES

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We prove the following theorem: Let $K$ be a connected symmetric subgroup of the group of isometries of a compact connected globally Riemannian symmetric space $M$. Then, $K$ acts in a variationally complete manner on $M$. (We refer to the work of R. Bott and H. Samelson [1; 2] for a definition of this concept and applications to topology.)

Let $G$ be a compact, connected Lie group, $L$ a connected symmetric subgroup of $G$. Bott and Samelson have proved the following: (1) $L$ acts in a variationally complete manner on $G/L$, the right coset space, (2) $L$ acts via the linear isotropy representation in a variationally complete way on the tangent space at a point of $G/L$.

Our theorem then generalizes (1), but does not imply (2). The method is essentially the same as Bott's [1], but uses a Lie algebra approach in a stronger way. The proof then provides an exposition of Bott's very important result from a Lie-algebra point of view. Notice also that it suggests the general program of studying the Jacobi-fields on a symmetric space as a problem of Lie algebra theory, even in the nonvariational complete case.

1. We briefly recall Bott's notations: If $p \in M$, $M_p$ denotes the tangent space to $M$ at $p$. Let $O_p(K)$ be the orbit of $K$ at $p$, let $O(K)_p$ be its tangent space at $p$, and let $g: R \to M$ be a geodesic of $M$ beginning at $p$, perpendicular to $O_p(K)$. Consider the vector space $J_p$ of Jacobi vector fields along $g$. A Jacobi field is a map $t \to Y_t \in M_p(t)$ for $t \in R$ that is a solution of the Jacobi Equation:

$$Y'' + R(X_t, Y_t)(X_t) = 0,$$

where $t \to X_t$ is the tangent vector field to the geodesic $g$, $R(X_t, Y_t)$ is the linear map $M_{g(t)} \to M_{g(t)}$ defined by the curvature form evaluated at $(X_t, Y_t)$ and $t \to Y_t''$ is the second covariant derivative of the field $Y_t$ along $g$.

$J^K_p$ denotes the focal subspace of $J_p$ relative to $O(K)$, i.e. $J^K_p$ consists of those fields $Y_t$ satisfying the initial conditions

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\[Y_0 \in O(K)_p,\]
\[V_0' + T_g(Y_0) \in O(K)_p,\]

where \(T_g\) is a certain linear transformation \(O(K)_p \to O(K)_p\). Hence, \(\dim J^K_g = \dim M\).

Let \(K\) denote the Lie algebra of \(K\). There is a linear mapping \(\pi: K \to J^K_g\) such that for \(k \in K\), \(\pi(k)\) is obtained by restricting the vector field determined by \(\mathfrak{a}\) on \(M\) to \(g\).

By definition, to prove variational completeness we must show that every element of \(J^K_g\) that vanishes at some point of \(g\) must lie in \(\pi(K)\).

2. \(M\) is a coset space \(G/L\), where \(L\) is a compact, symmetric subgroup of a compact, connected Lie group \(G\). It evidently suffices to prove the theorem in the case where \(p\) is the identity coset.

We have a natural reduction \(G = L \oplus M\), with \([M, M] \subseteq L\) and \([L, M] \subseteq M\). (For the ideas and results of the differential geometry of symmetric spaces, see [3].) \(M_p\) can be identified with \(M\), and each \(M_{\theta(t)}\), for \(t \in \mathbb{R}\), can be identified with \(M_p\) by parallel transport along \(g\). There is then a correspondence between vector fields along \(g\) and curves \(t \to Y_t\) in \(M\). The metric on \(M\) can be considered as induced by a positive definite quadratic form on \(G\) invariant under \(\text{Ad} G\).

Let \(P\) denote the projection of \(G\) on \(M\). Then \(N = P(K)\) is identified with \(O(K)_p\). Let \(X \in M\) correspond to \(X_0\). Then \(X \in N^\perp\). One sees that a curve \(t \to Y_t\) in \(M\) corresponds to a vector field in \(J^K_g\) if and only if

\[
Y_0 \in N, \quad Y'_0 + T_g(Y_0) \in N^\perp, \quad (2.2) \quad Y''_t = (\text{Ad} X)^2 Y_t,
\]

where \(t \to Y'_t\) is now the derivative in the ordinary sense. (One uses the explicit formula \(\text{Ad} [x, y] = -R(x, y)\) for the curvature in symmetric spaces and the fact that curvature is invariant under parallel translation in identifying \(R(X_t, Y_t)X_t\) with \((\text{Ad} X)^2 Y_t\), [3].)

If \(k \in K\), \(\pi(k)\) corresponds to the curve \(t \to \pi(k)\), \(= P((\text{Ad \ Exp} tX)(k))\), hence \(\pi(k) = P(k)\), \(\pi(k)' = P([X, k])\).

The question of variational completeness can now be treated in this Lie algebra setting as a property of solutions of vector-valued ordinary linear differential equations (2.2) with constant coefficients. In particular, a reduction of \(M\) into subspaces invariant under
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(Ad $X)^2$ leads to a decomposition of solutions of (2.2) into solutions taking values in the invariant subspaces.

3. The proof.

(a) kernel $\pi = K \cap L \cap \text{kernel Ad } X$.

(b) $\dim \pi(K) = \dim K - \dim K \cap L \cap \text{kernel Ad } X = \dim P(K) + \dim K \cap L - \dim K \cap L \cap \text{kernel Ad } X = \dim P(K) + \dim \text{Ad } X(K \cap L)$.

(c) $[K^\perp, K^\perp] \subset K$, $[K, K^\perp] \subset K^\perp$, since $K$ is a symmetric subalgebra of $G$. (The perpendicular operation $\perp$ is always with respect to the given metric on $G$.) $L^\perp = M$. $(\text{Ad } X)^2(P(K)) \subset P(K)$, since $X \in K^\perp \cap L^\perp$.

(d) $\text{Ad } X(K \cap L) \subset K^\perp \cap L^\perp$ hence $\text{Ad } X(K \cap L) \subset P(K)^\perp \cap M$. $(\text{Ad } X)^2(\text{Ad } X(K \cap L)) \subset (\text{Ad } X)^2(K^\perp \cap L^\perp) \subset \text{Ad } X(K \cap L)$.

(e) Define $Q = M \cap P(K)^\perp \cap \text{Ad } X(K \cap L)^\perp = (P(K) + \text{Ad } X(K \cap L))^\perp \cap M$. Then, $(\text{Ad } X)^2(Q) \subset Q$, $Q \subset K^\perp \cap L^\perp$.

(f) $\text{Ad}^2 X(Q) = 0$, for $\text{Ad } X(Q) \subset K \cap L$, hence $\text{Ad}^2 X(Q) \subset \text{Ad } X(K \cap L) \cap Q = 0$.

(g) $\dim M = \dim M = \dim J^K_q = \dim (\pi(K) + Q)$.

(h) If $k \in K$, then $\pi(k)_t \in Q^\perp$ for all $t \geq 0$.

Proof. $\pi(k)_t = P(\sinh(\text{Ad } tX)(k)) + P(\cosh(\text{Ad } tX)(k))$. Now, $P(\cosh(\text{Ad } tX)(k)) \subset P(K) \subset Q^\perp$. Then, $\sinh(\text{Ad } tX)(k) \in Q^\perp$ because of (e) and (f), and the fact that $(\text{Ad } X)^2$ is a symmetric transformation, with respect to the positive-definite quadratic form on $G$, that commutes with $P$.

Now, define a map $\psi: Q \to J^K_q$ as follows: For $q \in Q$, $\psi(q)_t$ is the curve in $M$ satisfying (2.2) and

\[
\psi(q_0) = 0, \\
\psi(q'_t) = q.
\]

Because of (e) and (f), $\psi(q)_t = tq$, and hence $\psi$ is one-to-one. Then $J^K_q = \psi(Q) + \pi(K)$, $\psi(Q) \cap \pi(K) = 0$ and the theorem follows. For if $t \to Y_t$ is an element of $J^K_q$, $Y_t = \pi(k)_t + tq$ for $k \subset K$, $q \in Q$, and $Y_t = 0$ for some $t > 0$, then $q = 0$ by (h).

Bibliography


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