THE UNIFORMIZATION OF A CLASS OF SIMPLY CONNECTED RIEMANN SURFACES

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After defining a class of simply connected Riemann surfaces, this study determines the type of a surface from this class and obtains the forms of the uniformizing function and its derivative.

Definition of class of surfaces. Let \( \{a_n\}_{n=1}^{\infty} \) and \( \{b_n\}_{n=1}^{\infty} \) be two infinite sequences, and for each \( n \geq 1 \) let \( \{a_i(n)\}_{i=1}^{q_n} \) and \( \{b_i(n)\}_{i=1}^{q_n} \) be finite sequences such that for \( n \geq 1 \) and \( 1 \leq i \leq q_n - 1 \):

\[
0 < a_i(1) < b_i(1) < a_{i+1}(1) < b_{i+1}(1) < a_1;
\]

for \( n \) even and \( n \geq 2 \),

\[
a_{n-1} < b_{n-1} < b_i(n) < a_i(n) < b_{i+1}(n) < a_{i+1}(n) < b_n < a_n;
\]

for \( n \) odd and \( n \geq 3 \),

\[
b_{n-1} < a_{n-1} < a_i(n) < b_i(n) < a_{i+1}(n) < b_{i+1}(n) < a_n < b_n.
\]

A surface \( F \) of the class to be discussed consists of sheets \( S_n \) and \( S_i(n) \), \( n = 1, 2, 3, \ldots, i = 1, 2, \ldots, q_n \), over the \( w \)-sphere where, for \( S_n \) and \( S_i(n) \) copies of the \( w \)-sphere:

(a) \( S_1 \) is slit along the real axis from \( a_1(1) \) to \( b_1(1) \), \( i = 1, 2, \ldots, q_1 \), and from \( a_1 \) to \( b_1 \). (b) for \( n \geq 1 \), \( S_{2n} \) is slit along the real axis from \( a_{2n-1} \) to \( b_{2n-1} \), from \( b_{2n} \) to \( a_{2n} \), and from \( b_i(2n) \) to \( a_i(2n) \), \( i = 1, 2, \ldots, q_{2n} \). (c) for \( n \geq 1 \), \( S_{2n+1} \) is slit along the real axis from \( b_{2n} \) to \( a_{2n} \), from \( a_{2n+1} \) to \( b_{2n+1} \), and from \( a_i(2n+1) \) to \( b_i(2n+1) \), \( i = 1, 2, \ldots, q_{2n+1} \). (d) for \( n \geq 1 \) and \( i = 1, 2, \ldots, q_n \), \( S_i(n) \) is slit along the real axis between \( a_i(n) \) and \( b_i(n) \). (e) \( S_n \) and \( S_{n+1} \) are joined along the slits between \( a_n \) and \( b_n \) to form first order branch points at the end points. (f) \( S_i(n) \) is joined to \( S_n \) along the slits between \( a_i(n) \) and \( b_i(n) \) to form first order branch points at the end points.

A special case of this class was considered in [2]. However, only the type of a surface was established.

The uniformizing function. \( F \) is simply connected, and by the Fundamental Mapping Theorem there exists a unique function \( g \) which maps \( F \) in a one-to-one way onto \( |z| < R \leq \infty \), and where for \( f(z) = g^{-1}(z) \), \( f(0) = 0 \in S_1 \) and \( f'(0) = 1 \). By slitting each sheet of \( F \) along the uncut part of the real axis, two surfaces of hyperbolic type are obtained, and an application of the reflection principle to the

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uniformizing function of one of these surfaces in the upper half plane shows that \( f(z) \) is real for \( z \) real. Furthermore, the image of \( F \) in the \( z \)-plane satisfies the following properties. The image of \( S_n \) or \( S_i(n) \) is a region symmetric about the real axis. \( S_1 \) and \( S_i(1), i = 1, 2, \ldots, q_1 \), are mapped onto a domain containing \( z = 0 \) and bounded by a simple closed curve \( C_1 \) which intersects the real axis at \( \alpha_1 \) and \( -\beta_1 \). For \( n > 1 \), \( S_n \) and \( S_i(n), i = 1, 2, \ldots, q_n \), are mapped onto an annular region about \( z = 0 \) bounded by two simple closed curves \( C_{n-1} \) and \( C_n \) which intersect the real axis at \( -\beta_{n-1} \) and \( \alpha_{n-1} \), and \( -\beta_n \) and \( \alpha_n \), respectively. Each \( S_i(n) \) is mapped onto a domain bounded by a simple closed curve \( C_i(n) \) which intersects the real axis at \( (-1)^{n+1} \beta_i(n) \) and \( (-1)^{n+1} \alpha_i(n) \). If \( (-1)^n \gamma_n \) and \( (-1)^{n+1} \delta_i(n) \) denote the image in the \( z \)-plane of \( w = \infty \) for \( S_n \) and \( S_i(n) \), while \( (-1)^n \delta_n \) and \( (-1)^{n+1} \delta_i(n) \) denote the image of \( w = 0 \) for \( S_n \) and \( S_i(n) \), respectively, then the following inequalities are satisfied:

for \( n \) even, \( n \geq 2 \),

\[
0 < \alpha_{n-1} < \delta_n < \gamma_n < \alpha_n < \alpha_{n+1};
\]

for \( n \) odd, \( n \geq 3 \),

\[
0 > -\gamma_1 > -\beta_1 > -\beta_{n-1} > -\delta_n > -\gamma_n > -\beta_n > -\beta_{n+1};
\]

for \( 1 \leq i \leq q_1 - 1 \),

\[
0 < \alpha_i(1) < \delta_i(1) < \gamma_i(1) < \beta_i(1) < \alpha_{i+1}(1) < \beta_{i+1}(1) < \alpha_1;
\]

for \( n \) even, \( n \geq 2 \), \( 1 \leq i \leq q_n - 1 \),

\[
-\beta_{n-1} > -\beta_i(n) > -\delta_i(n) > -\gamma_i(n) > -\alpha_i(n) > -\beta_{i+1}(n) > -\alpha_{i+1}(n) > -\beta_n;
\]

for \( n \) odd, \( n \geq 3 \), \( 1 \leq i \leq q_n - 1 \),

\[
\alpha_{n-1} < \alpha_i(n) < \delta_i(n) < \gamma_i(n) < \beta_i(n) < \alpha_{i+1}(n) < \beta_{i+1}(n) < \alpha_n.
\]

The approximating closed surfaces. Let \( F_n \) be the part of \( F \) formed from \( S_k, k = 1, 2, \ldots, 2n+1 \), and \( S_i(k), k = 1, 2, \ldots, 2n \) and \( i = 1, 2, \ldots, q_{2n} \), with the slits in \( S_{2n+1} \) from \( a_{2n+1} \) to \( b_{2n+1} \) and from \( a_i(n+1) \) to \( b_i(2n+1) \), \( i = 1, 2, \ldots, q_{2n+1} \), deleted.

Notation:

\[
\begin{align*}
\alpha_\phi &= 1 - z/\alpha_\phi, & \alpha_\phi^*(k) &= 1 + (-1)^k z/\alpha_\phi(k). \\
\beta_\phi &= 1 + z/\beta_\phi, & \beta_\phi^*(k) &= 1 + (-1)^k z/\beta_\phi(k). \\
\gamma_\phi &= 1 - (-1)^k z/\gamma_\phi, & \gamma_\phi^*(k) &= 1 + (-1)^k z/\gamma_\phi(k). \\
\delta_\phi &= 1 - (-1)^k z/\delta_\phi, & \delta_\phi^*(k) &= 1 + (-1)^k z/\delta_\phi(k).
\end{align*}
\]
Lemma 1.

\[ R_n(z) = \prod_{k=1}^{2n} \left[ \frac{\theta_{k,n}}{\gamma_{k,n}} \right] \prod_{i=1}^{q_k} \left[ \frac{\delta_{i,n}(k)}{\gamma_{i,n}(k)} \delta_{2n+1,n} \right] \]

where

\[ R_n'(z) = \prod_{k=1}^{2n} \left[ \frac{\alpha_{k,n} \beta_{k,n}}{\gamma_{k,n}} \prod_{i=1}^{q_k} \frac{\alpha_{i,n}(k) \beta_{i,n}(k)}{\gamma_{i,n}(k)} \right], \]

defines a mapping of the \( z \)-plane onto \( F_n \).

Proof. \( F_n \) is a simply connected closed surface and hence is the Riemann surface of the inverse of a unique rational function \( R_n \) such that \( w = R_n(z) \), \( R_n(0) = 0 \in S_1 \), \( R_n'(0) = 1 \), and \( R_n(\infty) = \infty \in S_{2n+1} \).

Lemma 2. \( F \) is parabolic.

Proof. Suppose that \( F \) is not parabolic, and thus \( g \) is a mapping of \( F \) onto \( \{ |z| < R < \infty \} \). Let \( D_n \) be the \( z \)-plane slit along the real axis from \( \alpha_{2n,2n} \) to \( +\infty \). \( \zeta = \psi_n(z) = g[R_n(z)] \) defines a schlicht mapping of \( D_n \) onto a simply connected region \( \Delta_n \) of the \( \zeta \)-plane bounded by \( C_{2n+1}, C_i(2n+1), i = 1, 2, \ldots, q_{2n+1} \), and the segments \( (\alpha_{2n,2n}, -\gamma_{2n}) \) and \( (-\beta_{2n+1}, -\gamma_{2n}) \). Then \( \alpha_{2n,2n} \leq d(0, C_{2n+1}) \leq R < \infty \) where \( d(0, C_{2n+1}) \) is the distance from \( \zeta = 0 \) to the curve \( C_{2n+1} \), and thus there exists a subsequence \( \{ \alpha_{2n,j,2n} \} \) such that \( \alpha_{2n,j,2n} \to A \leq R \). Then \( \psi_n \) is a schlicht mapping of \( D_n \) into \( \Delta_n \), where for \( n \) sufficiently large, \( \Delta_n \subset \{ |\zeta| \leq R \} \). If \( D \) is the \( z \)-plane slit along the real axis from \( z = A \) to \( +\infty \), then \( \{ \psi_n \} \) forms a normal family by Montel's Theorem and \( \psi_n(z) \to \psi(z) \) uniformly on any compact subset of \( D \). \( \psi \) maps \( D \) schlichtly onto \( \{ |\zeta| \leq R \} \). Then \( R_n(z) \to H(z) = f[\psi(z)] \), where \( H \) is a function meromorphic in \( D \). \( H(z) \neq \infty \) because \( R_n(0) = 0 \).

Let \( D^* \) be the \( z \)-plane slit along the real axis from \( z = A \) to \( -\infty \). Now \( R_n(z) \) is defined in \( D^* \) and for \( j \) sufficiently large assumes no negative real values in any compact subset of \( D^* \). Then \( \{ R_{nj} \} \) is a normal family in \( D^* \), so that a subsequence \( \{ R_m \} \) of \( \{ R_{nj} \} \) converges uniformly on any compact subset of \( D^* \) to a function \( G \). Because of a common domain of convergence, \( G \) is the analytic continuation of \( H \). Then \( w = G(z) \) defines a mapping of the \( z \)-plane punctured at \( z = A \) and \( \infty \) one to one and conformally onto an open doubly connected Riemann surface \( F^* \) of which \( F \) is a subsurface obtained by inserting some slits in \( F^* \) over the real axis. Clearly this is impossible, and thus \( R = \infty \).

Lemma 3. \( R_n(z) \to f(z) \) uniformly on any compact subset of \( |z| < \infty \) as \( n \to \infty \).
Proof. Because $F$ is parabolic and thus $\{\Delta_n\}$ converges in the sense of Carathéodory to its kernel $|\zeta| < \infty$, it follows that $z = R_n^{-1}[f(\zeta)] \rightarrow \xi = g[R_n(z)]$ uniformly on any compact subset of $|\zeta| < \infty$. Then $\{D_n\}$ converges to $|z| < \infty$ and thus $R_n(z) \rightarrow f(z)$ uniformly on any compact subsets of $|z| < \infty$.

Lemma 4. $\alpha_{k,n} \rightarrow \alpha_k$, $\beta_{k,n} \rightarrow \beta_k$, $\gamma_{k,n} \rightarrow \gamma_k$, $\delta_{k,n} \rightarrow \delta_k$, $\alpha_{i,n}(k) \rightarrow \alpha_i(k)$, $\beta_{i,n}(k) \rightarrow \beta_i(k)$, $\gamma_{i,n}(k) \rightarrow \gamma_i(k)$, and $\delta_{i,n}(k) \rightarrow \delta_i(k)$ as $n \rightarrow \infty$.

Proof. This is a consequence of Hurwitz’s Theorem.

Lemma 5. The infinite product
$$\pi(z) = [z/\gamma_1] \left[ \prod_{k=2}^{\infty} \delta_k/\gamma_k \right] \prod_{i=1}^{\infty} \delta_i^{*}(k)/\gamma_i^{*}(k)$$
converges uniformly on any compact subset of $|z| < \infty$.

Proof. $\pi(z)$ can be decomposed onto four infinite products similar to $M(z) = \prod_{k=1}^{\infty} \delta_2k/\gamma_2k$. Since $F$ is parabolic and $\{\Delta_n\} \rightarrow \{ |\zeta| < \infty \}$, then $\delta_k \rightarrow \infty$ and $\gamma_k \rightarrow \infty$ as $k \rightarrow \infty$. Therefore, for any $R > 0$ there exists $n_0 = n_0(R)$ such that $\delta_k > R$ and $\gamma_k > R$ for $k \geq n_0$. Then consider $M_p(z) = \prod_{k=n_0}^{n_0+p} \delta_2k/\gamma_2k$. $M_p$ is holomorphic and $M_p(z) \neq 0$ for $|z| \leq R$. A sufficient condition for the uniform convergence of $M_p$ in a region as $p \rightarrow \infty$ is the uniform convergence of
$$\sum_{k=n_0}^{n_0+p} \log (\delta_2k/\gamma_2k)$$
in this region as $p \rightarrow \infty$, where the logarithm is the principal value. By the Cauchy criterion, this last series converges uniformly in $\{ |z| \leq R \}$ provided for $\epsilon > 0$ and for all $z$ such that $|z| \leq R$ there exists $N(\epsilon) > 0$ such that for $n > N(\epsilon)$ and for all $p > 0$,
$$\left| \sum_{k=n_0+p}^{n_0+n+p} \log (\delta_2k/\gamma_2k) \right| < \epsilon.$$

Because $\cdots \delta_2k < \gamma_2k < \delta_2k+2 < \gamma_2k+2, \cdots$, then for $m \geq 1$ and $p > 0$,
$$0 < \sum_{k=n_0+p}^{n_0+n+p} [(1/\delta_2k)^m - (1/\gamma_2k)^m] < (1/\delta_2n_0+2n)^m,$$
and thus
$$\left| \sum_{k=n_0+p}^{n_0+n+p} \log (\delta_2k/\gamma_2k) \right| = \left| \sum_{m=1}^{\infty} \left( \frac{R^m}{m} \right) \sum_{k=n_0+p}^{n_0+n+p} [(1/\delta_2k)^m - (1/\gamma_2k)^m] \right| \leq \sum_{m=1}^{\infty} (R^m/m) (1/\delta_2n_0+2n)^m \leq \sum_{m=1}^{\infty} (R/\delta_2n_0+2n)^m = R/(\delta_2n_0+2n - R).$$
Because $\delta_k \to \infty$ as $k \to \infty$, the required inequality is established. Hence $M(z)$ converges uniformly on any compact subset of the $z$-plane. Similarly, the other products into which $\pi(z)$ can be decomposed converge uniformly on any compact subset of the $z$-plane.

**Lemma 6.** $\pi(z) = f(z)$.

**Proof.** Because of Lemma 4, there exists $r > 0$ such that $R_n(z)/z \neq 0$ and $\pi(z)/z \neq 0$ for $|z| < r$. Because for $m \geq 1$,

$$0 < \sum_{k=1}^{\infty} \left[ \frac{1}{\delta_{2k}} - \frac{1}{\gamma_{2k}} \right] \leq \frac{1}{\delta_{2i}}^m$$

and

$$0 < \sum_{k=1}^{\infty} \left[ \frac{1}{\delta_{2k,n}} - \frac{1}{\gamma_{2k,n}} \right] < \frac{1}{\delta_{2i,n}}^m,$$

it follows with the aid of Lemma 4 that for $n_0 > 1$,

$$0 \leq \limsup_{n \to \infty} \left[ \sum_{k=1}^{n} \left[ \frac{1}{\delta_{2k,n}} - \frac{1}{\gamma_{2k,n}} \right] - \sum_{k=1}^{\infty} \left[ \frac{1}{\delta_{2k}} - \frac{1}{\gamma_{2k}} \right] \right]$$

$$\leq \limsup_{n \to \infty} \left[ \sum_{k=n_0}^{n} \left[ \frac{1}{\delta_{2k,n}} - \frac{1}{\gamma_{2k,n}} \right] - \sum_{k=n_0}^{\infty} \left[ \frac{1}{\delta_{2k}} - \frac{1}{\gamma_{2k}} \right] \right]$$

$$< \lim_{n \to \infty} \left[ \frac{1}{\delta_{2n_0,n}} + \frac{1}{\delta_{2n_0}} \right] = 2/(\delta_{2n_0})^m.$$

Since $\delta_{2n_0} \to \infty$ as $n_0 \to \infty$,

$$\lim_{n \to \infty} \left\{ \sum_{k=1}^{n} \left[ \frac{1}{\delta_{2k,n}} - \frac{1}{\gamma_{2k,n}} \right] - \sum_{k=1}^{\infty} \left[ \frac{1}{\delta_{2k}} - \frac{1}{\gamma_{2k}} \right] \right\} = 0.$$

Thus in this manner all the coefficients of the Taylor expansion of $\log [R_n(z)/\pi(z)]$ about $z = 0$ can be shown to have a limit of zero as $n \to \infty$, and thus $\lim_{n \to \infty} \log [R_n(z)/\pi(z)] = 0$ which implies that $\lim_{n \to \infty} R_n(z) = f(z) = \pi(z)$.

**Lemma 7.** The infinite product

$$P(z) = \prod_{k=1}^{\infty} \left\{ \alpha_k^* \beta_k^* / (\gamma_k^*)^2 \right\} \left\{ \prod_{i=1}^{q_k} \alpha_i^* (k) \beta_i^* (k) / [\gamma_i^* (k)]^2 \right\}$$

converges uniformly on any compact subset of the $z$-plane.

**Proof.** $P(z)$ can be decomposed into four infinite products similar to $Q(z) = \prod_{k=1}^{\infty} \alpha_{2k}^* \alpha_{2k+1}^* / (\gamma_{2k+2})^2$. Because for $m \geq 1$ and $p > 0$,

$$0 < \sum_{k=m_0+p}^{m_0+m+p} \left[ \frac{1}{(\alpha_{2k})^m} + \frac{1}{(\alpha_{2k+1})^m} - \frac{1}{(\gamma_{2k+2})^m} \right] < 2/(\alpha_{2n_0+2n})^m,$$
an application of the Cauchy criterion similar to that described in the proof of Lemma 5 establishes the convergence of $Q(z)$. Similarly, the other factors of $P(z)$ converge.

**Lemma 8.** $P(z) = f'(z)$.

**Proof.** Because for all $i > 0$ and $m \geq 1$,

$$0 < \sum_{k=i}^{n} (1/\alpha_{2k})^m + \sum_{k=i}^{n-1} (1/\alpha_{2k+1,n})^m - \sum_{k=i}^{n-1} 2/(\gamma_{2k+2,n})^m < 2/(\alpha_{2i,n})^m,$$

$$0 < \sum_{k=i}^{\infty} [(1/\alpha_{2k})^m + (1/\alpha_{2k+1})^m - 2/(\gamma_{2k+2})^m] < 2/(\alpha_{2i})^m,$$

and similar inequalities, it follows in a manner similar to the proof of Lemma 6 that $\lim_{n \to \infty} \log [R_n'(z)/P(z)] = 0$ and thus $P(z) = f'(z)$.

These lemmas prove the following theorem.

A Riemann surface of the class defined is parabolic and its mapping function $f$ is given by

$$f(z) = [z/\gamma_1] \prod_{k=2}^{\infty} \left[ \delta_k^* / \gamma_k^* \right] \prod_{i=1}^{q_k} \delta_i^*(k) / \gamma_i^*(k)$$

where

$$f'(z) = \prod_{k=1}^{\infty} \left\{ [\alpha_k^* \beta_k^* / (\gamma_k^*)^2] \left[ \prod_{i=1}^{q_k} \alpha_i^* (k) \beta_i^* (k) / (\gamma_i^*)^2 \right] \right\}.$$