NEIGHBORHOOD EXTENSIONS OF CONTINUOUS SELECTIONS

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Hanner (see [2, Theorem 19.2]) has proved that the property of being a neighborhood extension space with respect to the class of paracompact spaces is a local property. Theorem 1 of this paper generalizes Hanner's result, and states roughly that if a continuous selection can be locally extended to a continuous selection, then it has a neighborhood extension which is also a continuous selection. The proof of Theorem 1 differs considerably from Hanner's proof, and seems at least as simple.

Let X be a paracompact space and Y a topological space. We assume that for each \( x \in X \), \( F(x) \) is a subset of \( Y \). If \( K \) is a subset of \( X \) and \( \phi \) is a mapping (continuous function) on \( K \) into \( Y \), then \( \phi \) is a selection for \( F \) if \( \phi(x) \in F(x) \) for each \( x \in K \). A subset \( S_1 \) of \( Y \) is an \( F \)-neighborhood extension set with respect to a subset \( S_2 \) of \( X \) if each mapping \( \phi: C \rightarrow S_1, \phi \) a selection for \( F \), \( C \subset S_2 \) and \( C \) closed relative to \( S_2 \), can be extended to a mapping \( \psi: G \rightarrow S_1 \), where \( \psi \) is a selection for \( F \), \( G \subset S_2 \) and \( G \) is open relative to \( S_2 \).

We assume that \( A \) is a closed subset of \( X \) and that \( f \) is a selection for \( F \) which maps \( A \) into \( Y \).

**Theorem 1.** If for each \( x \in A \) there exist neighborhoods \( V_x \) of \( x \) and \( U_x \) of \( f(x) \) such that \( U_x \) is an \( F \)-neighborhood extension set with respect to \( V_x \), and \( f[V_x \cap A] \subset U_x \), then there exists an open set \( N \supset A \) and a mapping \( g: N \rightarrow Y \) such that \( g|A = f \) and \( g \) is a selection for \( F \).

**Proof.** Let \( g_1 = \{ W \mid W = X - A \text{ or } W = V_x \text{ for some } x \in A \} \). Since \( X \) is paracompact, there is a \( \sigma \)-discrete refinement \( g_2 \) of \( g_1 \). We may assume without loss of generality that \( \{ W \mid W \in g_2 \} \) is also a refinement of \( g_1 \). We let \( g = \{ W \mid W \in g_2 \text{ and } W \cap A \neq \emptyset \} \). It is possible to represent \( g = \bigcup_{\alpha \in A} g_\alpha \), where each \( g_\alpha \) is a discrete collection of open sets.

We wish to define inductively a sequence \( A_0 \subset A_1 \subset A_2 \subset \cdots \) of closed sets and mappings \( g_k: A_k \rightarrow Y \) such that:

(i) \( A_0 = A \) and \( g_0 = g \),
(ii) if \( W \in g_k \), then \( W \cap A \) is contained in the interior of \( A_{k+1} \).

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(iii) each $g_k$ is a selection for $F$ and $g_{k+1}|A_k = g_k$ for $k = 0, 1, 2, \ldots$.

We define $A_0$ and $g_0$ so as to satisfy (i). Now let us assume that $A_0 \subset A_1 \subset \cdots \subset A_n$ and $g_0, g_1, \ldots, g_n$ have been defined so as to satisfy the above conditions. We must define $A_{n+1}$ and $g_{n+1}$.

Let $W$ be any member of $\mathcal{R}_s$. There exists $x \in A$ such that $W \cap V_x$. The set $g_n^{-1}[U_x]$ is open relative to $A_n$, and thus there exists an open set $W_1$ such that $g_n^{-1}[U_x] = A_n \cap W_1$. Since $g_n[\overline{W} \cap A] = f[\overline{W} \cap A] \subset f[\overline{V_x} \cap A] \subset U_x$, it follows that $\overline{W} \cap A \subset W_1$. By normality of $X$, there exists an open set $W_2$ such that $\overline{W} \cap A \subset W_2$ and $W_2 \subset W_1$. If we define $W_3 = W \cap W_2$, then $g_n[\overline{W_3} \cap A_n] \subset U_x$ and we may extend $g_n[\overline{W_3} \cap A_n]$ to a mapping $\psi_W : W_4 \to U_2$, where $W_4$ is an open subset of $\overline{V_x}$ and $\psi_W$ is a selection for $F$. There exists an open set $W_5$ such that $\overline{W_3} \cap A_n \subset W_5$ and $\overline{W_5} \subset W_4$.

We now define

$$A_{n+1} = A_n \cup \bigcup_{W \in \mathcal{R}_s} (\overline{W_5} \cap \overline{W_2}),$$

and

$$g_{n+1}(x) = \begin{cases} g_n(x), & \text{for } x \in A_n, \\ \psi_W(x), & \text{for } x \in \overline{W_5} \cap \overline{W_3} \text{ for some } W \in \mathcal{R}_s. \end{cases}$$

The fact that $\mathcal{R}_s$ is discrete implies that the set $A_{n+1}$ is closed. It is easy to see that for each $W \in \mathcal{R}_s$, $W \cap A \subset W_5 \cap W_6$ and hence $W \cap A$ is contained in the interior of $A_{n+1}$. The function $g_{n+1}$ is well defined, and its continuity follows from the fact that $\mathcal{R}_s$ is a discrete collection of open sets. It is also obvious that $g_{n+1}$ is a selection for $F$.

We now obtain $N$ and $g$ by letting $N = \bigcup_{n=1}^{\infty} (\text{interior } A_n)$, and defining $g(x) = g_n(x)$ for $x \in (\text{interior } A_n)$.

The preceding theorem is of interest in connection with the problem (Q2) stated by Michael in [4]. The next theorem, which is a combination lifting and extension theorem, is of the same general nature as Steenrod's existence theorem on cross sections (see [5, p. 55]). A special case of this result was used by the author in [1].

We now let $Y$ be a fiber space (in the sense of Hilton, see [3, p. 46]) with base space $B$, fiber $K$, and projection $p : Y \to B$. It is again assumed that $X$ is a paracompact space and that $A$ is a closed subset of $X$. We assume mappings $f : A \to Y$ and $h : X \to B$ such that $pf = h|A$. The definition of ANR (absolute neighborhood retract) is given in [2].

**Theorem 2.** If $K$ is an ANR, then there exists an open set $N \supset A$ and a mapping $g : N \to Y$ such that $g|A = f$ and $pg = h|N$. 

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Proof. We define $F(x) = p^{-1}h(x)$ for each $x \in X$. For each $x \in A$, let $W_x$ be a neighborhood of $h(x)$ such that there exists a homeomorphism $\eta_x$ on $W_x \times K$ onto $p^{-1}[W_x]$ for which $p \eta_x(b, k) = b$. We define $U_x = Y$, and choose $V_x$ to be any neighborhood of $x$ for which $h[V_x] \subset W_x$.

We will show that each $U_x$ is an $F$-neighborhood extension set with respect to $V_x$. Thus, we consider a closed subset $C$ of $V_x$ and a selection for $F$, $\phi: C \to U_x$. The mapping $\eta_x^{-1} \phi$ maps $C$ into $W_x \times K$, and hence we can represent $\eta_x^{-1} \phi(t) = (\alpha(t), \beta(t))$, where $\alpha$ and $\beta$ are mappings on $C$ into $W_x$ and $K$ respectively. Since $K$ is an ANR, and $V_x$ is normal we can extend $\beta$ to a mapping $\beta': G_1 \to K$, where $G_1$ is open relative to $V_x$. Since $\alpha(t) = p \eta_x(\alpha(t), \beta(t)) = p \eta_x \eta_x^{-1} \phi(t) = p \phi(t) = h(t)$ for $t \in C$, we can let $G_2 = V_x \cap h^{-1}[W_x]$ and extend $\alpha$ to $\alpha': G_2 \to W_x$ by letting $\alpha' = h|G_2$. We now obtain the desired selection for $F$ by letting $G = G_1 \cap G_2$ and defining

$$\psi(t) = \eta_x((\alpha'(t), \beta'(t))),$$

for $t \in G$.

It is now easy to verify that the hypotheses of Theorem 1 are satisfied. Thus, there exists an open set $N \supset A$ and a mapping $g: N \to Y$ such that $g \upharpoonright A = f$ and $g$ is a selection for $F$. The fact that $g$ is a selection for $F$, however, means that $pg = h|N$, and this concludes the proof of Theorem 2.

A space $Y$ has the neighborhood extension property with respect to the class of all paracompact spaces if corresponding to each paracompact space $X$ and mapping $f$ of a closed subset of $X$ into $Y$ there is an extension of $f$ whose domain is an open set. We let $M$ be the set of all spaces $Y$ which have the neighborhood extension property with respect to the class of all paracompact spaces. The following theorem, which we prove by applying Theorem 1, seems to have been first demonstrated by Hanner (see [2, p. 340]).

Theorem 3. If each point in a space $Y$ has a neighborhood which belongs to $M$, then $Y$ belongs to $M$.

Proof. Let $X$ be a paracompact space, let $A$ be a closed subset of $X$, and let $f$ be a mapping of $A$ into $Y$.

We define $F(x) = Y$ for each $x \in A$. For each $x \in A$, we choose a neighborhood $U_x$ of $f(x)$ such that $U_x \in M$. We may now choose a neighborhood $V_x$ of $x$ such that $f[V_x \cap A] \subset U_x$. Since $V_x$ is a paracompact space and $U_x \in M$, it follows that $U_x$ is an $F$-neighborhood extension set with respect to $V_x$. Thus, by Theorem 1 we can extend $f$ to a mapping whose domain is an open set. This proves that $Y \in M$. 

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A REMARK ON THE CHARACTERIZATION OF HOMOTHETIC TRANSFORMATION AND INVERSION

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Given two closed orientable $C^2$ surfaces $S, \bar{S}$ in $E^3$, let $h: S \rightarrow \bar{S}$ be a differentiable homeomorphism, such that referring to a suitable origin $O$

\[ \overline{X} = kX, \]

with $k$ nonzero and $C^2$ at points at which $X \neq 0, \overline{X} \neq 0$. Write

$\bar{p} = -X \cdot N, \bar{\bar{p}} = -\overline{X} \cdot \overline{N}$. Suppose further that $S, \bar{S}$ contains no pieces of cones with vertex $O$. We use the form $b = (\overline{N} \times X) \cdot dX$, introduced in [1]. We generalize results of [1] to

Theorem 1'. If $\int_S (H + k\overline{H}) \bar{p} dA = 0$, then $k = \text{const.}$, i.e. the map $h$ is a homothetic transformation with center $O$.

Proof. The formulae $\bar{p} d\overline{A} = k^4 \bar{p} dA$ and $db = (2/k^2) \bar{p} \overline{H} d\overline{A} + 2\overline{N} \cdot NdA$ yield $\int_S N \cdot \overline{N} dA = -\int_S kN \bar{H} dA$. Combining with $\int_S dA = \int_S dH \bar{p} dA$ gives

$$\int_S \int_S (1 \mp N \cdot \overline{N}) dA = \int_S \int_S (H \pm k\overline{H}) \bar{p} dA,$$

and the theorem follows as in [1].