ANOTHER PROOF OF THE MINIMAX THEOREM

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There are many known proofs of the fundamental theorem of 0-sum, 2 person game theory, the so-called minimax theorem. The following proof, however, seems to be the shortest yet.

If \( x = (x_1, x_2, \cdots, x_N) \) is a vector in \( \mathbb{R}^N \), then we write, as usual, \( x \geq 0 \) to mean \( x_i \geq 0 \), \( i = 1, \cdots, N \), \( x \geq 0 \) to mean \( X \geq 0 \) and \( X \neq 0 \). In what follows \( M \) denotes an arbitrary fixed real \( m \times n \) matrix and \( J \) denotes the \( m \times n \) matrix all of whose entries are 1. \( M^T \) denotes, as usual, the transpose of \( M \). Consider now,

1. **Minimax Theorem.** There exists a real number \( v \) such that
   \[
   (M - vJ)x \geq 0
   \]
   for some \( x \in \mathbb{R}^m \), \( x \geq 0 \) and
   \[
   ( - M^T + vJ^T )y \geq 0
   \]
   for some \( y \in \mathbb{R}^n \) with \( y \geq 0 \).

2. **Theorem of the Alternative.** Either
   \[
   Mx \geq 0 \quad \text{for some } x \in \mathbb{R}^m, \ x \geq 0,
   \]
   or
   \[
   - M^Ty \geq 0 \quad \text{for some } y \in \mathbb{R}^n, \ y \geq 0.
   \]

3. **Stiemke's Theorem** [1]. If \( S \) is a subspace of \( \mathbb{R}^N \) and \( S^\perp \) is its orthogonal complement, then \( S \cup S^\perp \) contains some vector \( X \) with \( X \geq 0 \).

We shall prove 3 and 3\( \rightarrow \)2\( \rightarrow \)1 (although the proofs of 3 and 2\( \rightarrow \)1 are standard we include them for completeness).

**Proof of 3.** Let \( A \) be the (closed) set of all vectors \( x \in \mathbb{R}^N \) such that \( \mid x \mid \geq 1 \), \( x \geq 0 \). Let \( P \) be the operator of projection onto \( S \), call \( B = P(A) \) and let \( y = P(z) \) be a vector in \( B \) of minimal length. Suppose \( y = (y_1, y_2, \cdots, y_n) \) had some negative component, say \( -y_i \), then, with \( w = (0, 0, \cdots, y_i, 0, \cdots) \), \( \mid y+w \mid < \mid y \mid \), and so \( \mid P(z+w) \mid = \mid y+P(w) \mid \leq \mid y+w \mid < \mid y \mid \), and this is a contradiction since \( Z+W \) clearly lies in \( A \). Hence \( y \geq 0 \). If \( y = 0 \) then \( z \in S^\perp \) and the result follows since \( z \in A \). If \( y \geq 0 \) then the result follows since \( y \in S \).

We now need the following

**Definition.** If \( (z_1, z_2, \cdots, z_m) \in \mathbb{R}^m \) and \( (w_1, w_2, \cdots, w_n) \) = \( w \in \mathbb{R}^n \) then the vector in \( \mathbb{R}^{m+n} \) given by
\[
(z_1z_2, \cdots z_m, w_1, w_2, \cdots w_n)
\]
will be denoted by \( z \times w \).

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Proof that $3 \rightarrow 2$. It is easily seen that the set of all vectors of the form $x \times Mx, x \in \mathbb{E}^n$, forms a subspace of $E^{n+n}$, as does the set of all vectors of the form $-M^T y \times y, y \in \mathbb{E}^n$. Next note that these subspaces are in fact orthogonal complements in $E^{n+n}$. An application of 3 tells us that either $x \times Mx \geq 0$ for some $X \in \mathbb{E}^n$ (in which case $x \geq 0, Mx \geq 0$) or that $-M^T y \times y \geq 0$ for some $y \in \mathbb{E}^n$ (in which case $y \geq 0, -M^T y \geq 0$). In either case 2 is verified.

Proof that $2 \rightarrow 1$. Let $S_1$ be the set of all real numbers, $\nu$, for which $(M - \nu J)x \geq 0$ for some $x \geq 0$ and similarly let $S_2$ be the set of $\nu$ for which $(-M^T + \nu J^T)y \geq 0$ for some $y \geq 0$. It follows directly that both $S_1$ and $S_2$ are closed. Neither $S_1$ nor $S_2$ are void since $S_1$ contains all large negative numbers while $S_2$ contains all large positive numbers. Applying 2 to the matrix $M - \nu J$ tells us that every $\nu \in S_1 \cup S_2$. Connectedness of the line now implies that $S_1$ and $S_2$ must overlap, and this is the statement 1.

Reference


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