ANOTHER PROOF OF THE MINIMAX THEOREM

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There are many known proofs of the fundamental theorem of 0-sum, 2 person game theory, the so-called minimax theorem. The following proof, however, seems to be the shortest yet.

If $x=(x_1, x_2, \cdots, x_N)$ is a vector in E^N, then we write, as usual, $x \geqq 0$ to mean $x_i \geqq 0$, $i=1, \cdots, N$, $x \geqq 0$ to mean $X \geq 0$ and $X \neq 0$. In what follows M denotes an arbitrary fixed real $m \times n$ matrix and J denotes the $m \times n$ matrix all of whose entries are 1. M^T denotes, as usual, the transpose of M. Consider now,

1. MINIMAX THEOREM. *There exists a real number ν such that $(M-\nu J)x \geqq 0$ for some $x \in E^m$, $x \geq 0$ and $(-M^T+\nu J^T)y \geqq 0$ for some $y \in E^n$ with $y \geq 0$.*

2. THEOREM OF THE ALTERNATIVE. *Either*

$$Mx \geqq 0 \text{ for some } x \in E^m,\ x \geq 0,$$

or

$$-M^T y \geqq 0 \text{ for some } y \in E^n,\ y \geq 0.$$

3. STIEMKE'S THEOREM [1]. *If S is a subspace of E^N and $S^\perp$ is its orthogonal complement, then $S \cup S^\perp$ contains some vector X with $X \geqq 0$.*

We shall prove 3 and 3→2→1 (although the proofs of 3 and 2→1 are standard we include them for completeness).

PROOF OF 3. Let A be the (closed) set of all vectors $x \in E^N$ such that $|x| \geqq 1$, $x \geqq 0$. Let P be the operator of projection onto S, call $B=P(A)$ and let $y=P(z)$ be a vector in B of minimal length. Suppose $y=(y_1, y_2 \cdots y_n)$ had some negative component, say $-y_i$, then, with $w=(0, 0, \cdots, y_i, 0, \cdots)$, $|y+w| < |y|$, and so $|P(z+w)| = |y+P(w)| \leqq |y+w| < |y|$, and this is a contradiction since $Z+W$ clearly lies in A. Hence $y \geqq 0$. If $y=0$ then $z \in S^\perp$ and the result follows since $z \in A$. If $y \geq 0$ then the result follows since $y \in S$.

We now need the following

DEFINITION. If $(z_1, z_2, \cdots, z_m)=Z \in E^m$ and $(w_1, w_2, \cdots, w_n) = w \in E^n$ then the vector in E^{m+n} given by

$$(z_1 z_2, \cdots z_m, w_1, w_2, \cdots w_n)$$

will be denoted by $z \times w$.

Received by the editors October 24, 1959 and, in revised form, January 4, 1960.

PROOF THAT 3→2. It is easily seen that the set of all vectors of the form $x \times Mx$, $x \in E^m$, forms a subspace of E^{m+n}, as does the set of all vectors of the form $-M^T y \times y$, $y \in E^n$. Next note that these subspaces are in fact orthogonal complements in E^{m+n}. An application of 3 tells us that either $x \times Mx \geqq 0$ for some $X \in E^m$ (in which case $x \geq 0$, $Mx \geqq 0$) or that $-M^T y \times y \geqq 0$ for some $y \in E^n$ (in which case $y \geq 0$, $-M^T y \geqq 0$). In either case 2 is verified.

PROOF THAT 2→1. Let S_1 be the set of all real numbers, ν, for which $(M-\nu J)x \geqq 0$ for some $x \geqq 0$ and similarly let S_2 be the set of ν for which $(-M^T+\nu J^T)y \geqq 0$ for some $y \geqq 0$. It follows directly that both S_1 and S_2 are closed. Neither S_1 nor S_2 are void since S_1 contains all large negative numbers while S_2 contains all large positive numbers. Applying 2 to the matrix $M-\nu J$ tells us that every $\nu \in S_1 \cup S_2$. Connectedness of the line now implies that S_1 and S_2 must overlap, and this is the statement 1.

REFERENCE

1. A. W. Tucker, *Extensions of theorems of Farkas and Stiemke*, Abstract 76, Bull. Amer. Math. Soc. vol. 56 (1950) p. 57.

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