THE ISOMETRIES OF SOME FUNCTION SPACES

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I. Introduction. For each positive real number p, H_p is the space of all functions which are analytic in the open unit disc and for which

$$\|f\|_p = \sup_{0<r<1} \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta\right]^{1/p} \tag{1.1}$$

is finite. If $p \geqq 1$, H_p is a complex Banach space, with (1.1) as norm.

The space of all bounded analytic functions in the open unit disc will be denoted by H_∞. It also is a complex Banach space, with the norm

$$\|f\|_\infty = \sup_{|z|<1} |f(z)|. \tag{1.2}$$

H_2 is a Hilbert space, and thus has many isometries; most of these do not preserve the additional structure which H_2 has as a consequence of the fact that it is a space of *analytic* functions. For H_1 and H_∞, however, the opposite is true: here the isometries are induced by conformal maps of the unit disc. We shall prove the following:

THEOREM 1. *Every linear isometry of H_∞ onto H_∞ is of the form*

$$(Tf)(z) = \alpha\cdot f(t(z)) \qquad (f \in H_\infty,\ |z| < 1), \tag{1.3}$$

where α is a complex number of absolute value 1 *and t is a conformal map of the unit disc onto itself.*

Conversely, if T satisfies (1.3), *it is a linear isometry of H_∞ onto H_∞.*

THEOREM 2. *Every linear isometry of H_1 onto H_1 is of the form*

$$(Tf)(z) = \alpha\cdot t'(z)\cdot f(t(z)), \qquad (f \in H_1;\ |z| < 1), \tag{1.4}$$

where α and t are as in Theorem 1. *Conversely, if T satisfies* (1.4), *it is a linear isometry of H_1 onto H_1.*

These theorems fail if T is not assumed to be *onto*, since multiplication by any Blaschke product is a linear isometry of H_p into H_p ($1 \leqq p \leqq \infty$). Also, Halsey Royden has pointed out that one cannot replace "linear" by "real-linear" in the hypotheses; for if $\bar{f}(z)$ is the

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complex conjugate of $f(\bar{z})$, then the map $f \to \tilde{f}$ is a real-linear isometry of H_p onto H_p $(1 \leqq p \leqq \infty)$.

Before we turn to the proofs of these results, we consider a more abstract situation.

II. **Isometries of sup-norm algebras.** A semi-simple commutative Banach algebra A with unit whose Gelfand representation is a uniformly closed subalgebra of $C(\Delta)$ (the space of all continuous functions on the maximal ideal space Δ of A) will be called a *sup-norm algebra*. We shall regard any such algebra as a uniformly closed subalgebra of $C(\Delta)$, equipped with the norm

$$\|f\| = \sup_{x \in \Delta} |f(x)| \qquad (f \in A). \tag{2.1}$$

THEOREM 3. *Every linear isometry of a sup-norm algebra A onto itself is of the form*

$$Tf = \alpha \cdot T_1 f \qquad (f \in A), \tag{2.2}$$

where $\alpha \in A$, $\alpha^{-1} \in A$, $|\alpha(x)| = 1$ for all $x \in \Delta$, and T_1 is an automorphism of A.

PROOF. To begin with, we regard A and $C(\Delta)$ merely as Banach spaces, and we let $S(A^*)$ and $S(C^*)$ denote the unit balls of their dual spaces A^* and C^*. If Y is the subspace of C^* which annihilates A, then there is a homomorphism h of C^* onto A^* whose kernel is Y. If L is an extreme point of $S(A^*)$, then $h^{-1}(L) \cap S(C^*)$ is nonempty (by the Hahn-Banach theorem), weak*-compact and convex and hence has an extreme point, by the Krein-Milman theorem; any such extreme point is also an extreme point of $S(C^*)$, and hence is of the form λL_x, with $|\lambda| = 1$. (We write L_x for the functional which assigns to each f the number $f(x)$.) We have proved:

To every extreme point L of $S(A^*)$ there corresponds a complex number λ of absolute value 1 and a point $x \in \Delta$ such that

$$Lf = \lambda f(x) \qquad (f \in A). \tag{2.3}$$

Furthermore, λ and x are uniquely determined by L, since A separates points on Δ.

Let ∂A be the set of all x obtained in this way. We note that A has the maximum modulus property with respect to ∂A:

To every $f \in A$ there corresponds a point $x \in \partial A$ such that $\|f\| = |f(x)|$.

Indeed, $\|f\| = \max |Lf|$ $(L \in S(A^*))$; the maximum is attained, since $S(A^*)$ is weak*-compact, and it is attained at an *extreme point* of $S(A^*)$.

The preceding considerations were suggested by Lemmas 4.3 and 6.1 of [1].

Now let T be a linear isometry of A onto A. Its adjoint T^* is an isometry of A^* onto A^*, and thus carries the set of extreme points of $S(A^*)$ onto itself. Hence to each $x\in\partial A$ there corresponds a complex number $\alpha(x)$, $|\alpha(x)|=1$, and a point $\phi(x)\in\partial A$, such that

$$T^*L_x = \alpha(x)L_\phi(x). \tag{2.4}$$

That is to say,

$$(Tf)(x) = L_xTf = T^*L_xf = \alpha(x)L_{\phi(x)}f \tag{2.5}$$

or

$$(Tf)(x) = \alpha(x)f(\phi(x)) \qquad (f\in A,\ x\in\partial A). \tag{2.6}$$

Taking $f=1$ in (2.6), we see that α coincides with $T1$ on ∂A. We extend the definition of α to Δ by setting $\alpha(x)=(T1)(x)$ for all $x\in\Delta$. Then $\alpha\in A$.

So far we have only used the linear structure of A. If we now substitute a product fg into (2.6), we obtain

$$(T(fg))(x) = \alpha(x)f(\phi(x))g(\phi(x)). \tag{2.7}$$

Multiplying by $\alpha(x)$, this becomes

$$\alpha(x)(T(fg))(x) = (Tf)(x)\cdot(Tg)(x) \qquad (f, g\in A;\ x\in\partial A), \tag{2.8}$$

by (2.6). Thus the function $\alpha\cdot T(fg)-(Tf)\cdot(Tg)$, which belongs to A, is 0 on ∂A, and hence is identically 0, by the maximum modulus property established above. Thus

$$\alpha\cdot T(fg) = (Tf)\cdot(Tg) \qquad (f, g\in A). \tag{2.9}$$

Choosing f and g so that $Tf=Tg=1$ (which is possible, since T is onto), (2.9) shows that $\alpha^{-1}\in A$. The maximum modulus property, applied to α and to α^{-1}, then implies that $|\alpha(x)|=1$ for all $x\in\Delta$.

Finally, put $T_1f=\alpha^{-1}Tf$. Then T_1 is a one-to-one linear map of A onto A which is also multiplicative, by (2.9), and this completes the proof.

III. **Proofs of Theorems 1 and 2.** H_∞ is a sup-norm algebra, as follows readily from the definition (1.2) of the norm. Its maximal ideal space Δ has a complicated structure, but all we need to know here is that Δ contains the open unit disc. Every automorphism of H_∞ is induced by a conformal map of the unit disc [3; 4, Theorem 9], and the constants are the only functions in H_∞ whose absolute value is constant. The direct part of Theorem 1 is thus a consequence of Theorem 3; the converse is trivial.

We note, in passing, that the same proof applies if H_∞ is replaced by the ring of all single-valued bounded analytic functions in a domain which is "maximal" in the sense in which this term was defined in [4].

Our proof of Theorem 2 leans heavily on some of the results obtained in [2], and we refer to that paper for the notion of "outer function" and for those properties of H_1 which we shall use.

In [2] it is proved that the extreme points of the unit ball of H_1 are precisely the outer functions of norm 1, and that the closure of this set of extreme points consists of all $f\in H_1$ with $\|f\|_1=1$ which have no zero in $|z|<1$.

If now T is an isometry of H_1 onto H_1, it follows that f has no zero in $|z|<1$ if and only if Tf has no zero either. If $k=T1$ and if λ is a complex number, the linearity of T shows that $T(f-\lambda)=Tf-\lambda k$. Since $f-\lambda$ has no zero if and only if $Tf-\lambda k$ has no zero, it follows that λ is in the range of f if and only if λ is in the range of $k^{-1}Tf$. We have proved:

For every $f\in H_1$, f and $k^{-1}Tf$ have the same range.

We set $Uf=k^{-1}Tf$. The restriction of U to H_∞ is then a linear isometry of H_∞ into H_∞. Moreover, if $b\in H_\infty$ and $f=T^{-1}(bk)$, then $f\in H^1$, and

$$Uf = k^{-1}Tf = b; \tag{3.1}$$

thus $k^{-1}Tf\in H_\infty$, and since f has the same range, $f\in H_\infty$. We have proved that U maps H_∞ *onto* H_∞. Theorem 1 applies to U and yields

$$(Tf)(z) = k(z)f(t(z)) \qquad (f\in H_\infty,\ |z| < 1), \tag{3.2}$$

since $U1=1$; here t is a conformal map of the unit disc onto itself. Next,

$$\|Tf\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |k(e^{i\theta})| \cdot |f(t(e^{i\theta}))|\,d\theta \tag{3.3}$$

for every $f\in H_\infty$, and

$$\|f\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |t'(e^{i\theta})| \cdot |f(t(e^{i\theta}))|\,d\theta. \tag{3.4}$$

To every bounded measurable real function u on $|z|=1$ there corresponds an $f\in H_\infty$ such that $|f|=e^u$ a.e. on $|z|=1$. Since T is an isometry of H_1, the last two terms of (3.3) and (3.4) are equal, for all $f\in H_\infty$, so that

$$\int_{-\pi}^{\pi} |k| e^{u} d\theta = \int_{-\pi}^{\pi} |t'| e^{u} d\theta \tag{3.5}$$

for every bounded measurable real u. This is possible only if $|k| = |t'|$ a.e. on $|z| = 1$. Since both k and t' are outer functions ($k = T1$, and 1 is outer!), it follows that $k = \alpha t'$, with $|\alpha| = 1$.

Thus (3.2) becomes (1.4), valid for all $f \in H_\infty$. Since H_∞ is dense in H_1, we see that (1.4) holds for all $f \in H_1$, and the direct part of Theorem 2 is proved. The converse is trivial.

ADDED IN PROOF. After completion of this paper, we noticed a paper by M. Nagasawa, *Isomorphisms between commutative Banach algebras with an application to rings of analytic functions*, Kōdai Math. Sem. Rep. vol. 11 (1959) pp. 182–188. Using the fact that sup-norm algebras may be regarded as operator algebras on a Hilbert space, Nagasawa obtains our Theorems 1 and 2.

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