THE ISOMETRIES OF SOME FUNCTION SPACES

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1. Introduction. For each positive real number $p$, $H_p$ is the space of all functions which are analytic in the open unit disc and for which

$$
(1.1) \quad \|f\|_p = \sup_{0<r<1} \left[ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right]^{1/p}
$$

is finite. If $p \geq 1$, $H_p$ is a complex Banach space, with (1.1) as norm.

The space of all bounded analytic functions in the open unit disc will be denoted by $H_\infty$. It also is a complex Banach space, with the norm

$$
(1.2) \quad \|f\|_\infty = \sup_{|z|<1} |f(z)|.
$$

$H_2$ is a Hilbert space, and thus has many isometries; most of these do not preserve the additional structure which $H_2$ has as a consequence of the fact that it is a space of analytic functions. For $H_1$ and $H_\infty$, however, the opposite is true: here the isometries are induced by conformal maps of the unit disc. We shall prove the following:

**Theorem 1.** Every linear isometry of $H_\infty$ onto $H_\infty$ is of the form

$$
(1.3) \quad (Tf)(z) = \alpha \cdot f(t(z)) \quad (f \in H_\infty, \ |z| < 1),
$$

where $\alpha$ is a complex number of absolute value 1 and $t$ is a conformal map of the unit disc onto itself.

Conversely, if $T$ satisfies (1.3), it is a linear isometry of $H_\infty$ onto $H_\infty$.

**Theorem 2.** Every linear isometry of $H_1$ onto $H_1$ is of the form

$$
(1.4) \quad (Tf)(z) = \alpha \cdot t'(z) \cdot f(t(z)), \quad (f \in H_1; \ |z| < 1),
$$

where $\alpha$ and $t$ are as in Theorem 1. Conversely, if $T$ satisfies (1.4), it is a linear isometry of $H_1$ onto $H_1$.

These theorems fail if $T$ is not assumed to be onto, since multiplication by any Blaschke product is a linear isometry of $H_p$ into $H_p$ ($1 \leq p \leq \infty$). Also, Halsey Royden has pointed out that one cannot replace "linear" by "real-linear" in the hypotheses; for if $f(z)$ is the
complex conjugate of $f(\bar{z})$, then the map $f \mapsto \bar{f}$ is a real-linear isometry of $H_p$ onto $H_p$ ($1 \leq p \leq \infty$).

Before we turn to the proofs of these results, we consider a more abstract situation.

II. Isometries of sup-norm algebras. A semi-simple commutative Banach algebra $A$ with unit whose Gelfand representation is a uniformly closed subalgebra of $C(\Delta)$ (the space of all continuous functions on the maximal ideal space $\Delta$ of $A$) will be called a sup-norm algebra. We shall regard any such algebra as a uniformly closed subalgebra of $C(\Delta)$, equipped with the norm

$$\|f\| = \sup_{x \in \Delta} |f(x)| \quad (f \in A).$$

**Theorem 3.** Every linear isometry of a sup-norm algebra $A$ onto itself is of the form

$$Tf = \alpha \cdot T_1 f \quad (f \in A),$$

where $\alpha \in A$, $\alpha^{-1} \in A$, $|\alpha(x)| = 1$ for all $x \in \Delta$, and $T_1$ is an automorphism of $A$.

**Proof.** To begin with, we regard $A$ and $C(\Delta)$ merely as Banach spaces, and we let $S(A^*)$ and $S(C^*)$ denote the unit balls of their dual spaces $A^*$ and $C^*$. If $Y$ is the subspace of $C^*$ which annihilates $A$, then there is a homomorphism $h$ of $C^*$ onto $A^*$ whose kernel is $Y$. If $L$ is an extreme point of $S(A^*)$, then $h^{-1}(L) \cap S(C^*)$ is nonempty (by the Hahn-Banach theorem), weak*-compact and convex and hence has an extreme point, by the Krein-Milman theorem; any such extreme point is also an extreme point of $S(C^*)$, and hence is of the form $\lambda L_x$, with $|\lambda| = 1$. (We write $L_x$ for the functional which assigns to each $f$ the number $f(x)$.) We have proved:

To every extreme point $L$ of $S(A^*)$ there corresponds a complex number $\lambda$ of absolute value 1 and a point $x \in \Delta$ such that

$$Lf = \lambda f(x) \quad (f \in A).$$

Furthermore, $\lambda$ and $x$ are uniquely determined by $L$, since $A$ separates points on $\Delta$.

Let $\partial A$ be the set of all $x$ obtained in this way. We note that $A$ has the maximum modulus property with respect to $\partial A$:

To every $f \in A$ there corresponds a point $x \in \partial A$ such that $\|f\| = |f(x)|$.

Indeed, $\|f\| = \max |Lf| (L \in S(A^*))$; the maximum is attained, since $S(A^*)$ is weak*-compact, and it is attained at an extreme point of $S(A^*)$. 
The preceding considerations were suggested by Lemmas 4.3 and 6.1 of [1].

Now let $T$ be a linear isometry of $A$ onto $A$. Its adjoint $T^*$ is an isometry of $A^*$ onto $A^*$, and thus carries the set of extreme points of $S(A^*)$ onto itself. Hence to each $x \in \partial A$ there corresponds a complex number $\alpha(x)$, $|\alpha(x)| = 1$, and a point $\phi(x) \in \partial A$, such that

\begin{equation}
T^*L_x = \alpha(x)L_{\phi(x)}.
\end{equation}

That is to say,

\begin{equation}
(Tf)(x) = L_x Tf = T^*L_x f = \alpha(x)L_{\phi(x)}f
\end{equation}

or

\begin{equation}
(Tf)(x) = \alpha(x)f(\phi(x)) \quad (f \in A, x \in \partial A).
\end{equation}

Taking $f = 1$ in (2.6), we see that $\alpha$ coincides with $T1$ on $\partial A$. We extend the definition of $\alpha$ to $\Delta$ by setting $\alpha(x) = (T1)(x)$ for all $x \in \Delta$. Then $\alpha \in A$.

So far we have only used the linear structure of $A$. If we now substitute a product $fg$ into (2.6), we obtain

\begin{equation}
(T(fg))(x) = \alpha(x)f(\phi(x))g(\phi(x)).
\end{equation}

Multiplying by $\alpha(x)$, this becomes

\begin{equation}
\alpha(x)(T(fg))(x) = (Tf)(x) \cdot (Tg)(x) \quad (f, g \in A; x \in \partial A),
\end{equation}

by (2.6). Thus the function $\alpha \cdot T(fg) - (Tf) \cdot (Tg)$, which belongs to $A$, is 0 on $\partial A$, and hence is identically 0, by the maximum modulus property established above. Thus

\begin{equation}
\alpha \cdot T(fg) = (Tf) \cdot (Tg) \quad (f, g \in A).
\end{equation}

Choosing $f$ and $g$ so that $Tf = Tg = 1$ (which is possible, since $T$ is onto), (2.9) shows that $\alpha^{-1} \in A$. The maximum modulus property, applied to $\alpha$ and to $\alpha^{-1}$, then implies that $|\alpha(x)| = 1$ for all $x \in \Delta$.

Finally, put $T_1 f = \alpha^{-1}Tf$. Then $T_1$ is a one-to-one linear map of $A$ onto $A$ which is also multiplicative, by (2.9), and this completes the proof.

III. Proofs of Theorems 1 and 2. $H_\infty$ is a sup-norm algebra, as follows readily from the definition (1.2) of the norm. Its maximal ideal space $\Delta$ has a complicated structure, but all we need to know here is that $\Delta$ contains the open unit disc. Every automorphism of $H_\infty$ is induced by a conformal map of the unit disc [3; 4, Theorem 9], and the constants are the only functions in $H_\infty$ whose absolute value is
constant. The direct part of Theorem 1 is thus a consequence of Theorem 3; the converse is trivial.

We note, in passing, that the same proof applies if $H_\omega$ is replaced by the ring of all single-valued bounded analytic functions in a domain which is "maximal" in the sense in which this term was defined in [4].

Our proof of Theorem 2 leans heavily on some of the results obtained in [2], and we refer to that paper for the notion of "outer function" and for those properties of $H_1$ which we shall use.

In [2] it is proved that the extreme points of the unit ball of $H_1$ are precisely the outer functions of norm 1, and that the closure of this set of extreme points consists of all $f \in H_1$ with $\|f\|_1 = 1$ which have no zero in $|z| < 1$.

If now $T$ is an isometry of $H_1$ onto $H_1$, it follows that $f$ has no zero in $|z| < 1$ if and only if $Tf$ has no zero either. If $k = T1$ and if $\lambda$ is a complex number, the linearity of $T$ shows that $T(f - \lambda) = Tf - \lambda k$. Since $f - \lambda$ has no zero if and only if $Tf - \lambda k$ has no zero, it follows that $\lambda$ is in the range of $f$ if and only if $\lambda$ is in the range of $k^{-1}Tf$. We have proved:

For every $f \in H_1$, $f$ and $k^{-1}Tf$ have the same range.

We set $Uf = k^{-1}Tf$. The restriction of $U$ to $H_\omega$ is then a linear isometry of $H_\omega$ into $H_\omega$. Moreover, if $b \in H_\omega$ and $f = T^{-1}(bk)$, then $f \in H_1$, and

$$Uf = k^{-1}Tf = b;$$

thus $k^{-1}Tf \in H_\omega$, and since $f$ has the same range, $f \in H_\omega$. We have proved that $U$ maps $H_\omega$ onto $H_\omega$. Theorem 1 applies to $U$ and yields

$$Tf(z) = k(z)f(t(z)) \quad (f \in H_\omega, \quad |z| < 1),$$

since $U1 = 1$; here $t$ is a conformal map of the unit disc onto itself.

Next,

$$\|Tf\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |k(e^{i\theta})| \cdot |f(t(e^{i\theta}))| \, d\theta$$

for every $f \in H_\omega$, and

$$\|f\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})| \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(e^{i\theta})| \cdot |f(t(e^{i\theta}))| \, d\theta.$$

To every bounded measurable real function $u$ on $|z| = 1$ there corresponds an $f \in H_\omega$ such that $|f| = e^u$ a.e. on $|z| = 1$. Since $T$ is an isometry of $H_1$, the last two terms of (3.3) and (3.4) are equal, for all $f \in H_\omega$, so that
for every bounded measurable real \( u \). This is possible only if \( |k| = |t'| \) a.e. on \( |z| = 1 \). Since both \( k \) and \( t' \) are outer functions (\( k = T_1 \), and 1 is outer!), it follows that \( k = \alpha t' \), with \( |\alpha| = 1 \).

Thus (3.2) becomes (1.4), valid for all \( f \in H_\infty \). Since \( H_\infty \) is dense in \( H_1 \), we see that (1.4) holds for all \( f \in H_1 \), and the direct part of Theorem 2 is proved. The converse is trivial.

**Added in proof.** After completion of this paper, we noticed a paper by M. Nagasawa, *Isomorphisms between commutative Banach algebras with an application to rings of analytic functions*, Kodai Math. Sem. Rep. vol. 11 (1959) pp. 182–188. Using the fact that sup-norm algebras may be regarded as operator algebras on a Hilbert space, Nagasawa obtains our Theorems 1 and 2.

**References**