SIMPLE BOUNDS FOR BURNSIDE p-GROUPS

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1. The Burnside p-group $B(p^r, n)$, where p is a prime number, is the group generated by n elements $a_1, a_2, \cdots, a_n$ subject only to the defining relations

$$x^{p^r} = 1$$

for every element x in the group.

The purpose of the following note is to show how some bounds for the orders of such groups may be obtained quite simply. §2, which concerns an upper bound for $B(3, n)$, is prompted by a remark in the recent book by Marshall Hall on group theory [2]. §3 gives a lower bound[1] for the order of $B(p^r, n)$ which generalizes a result of Sanov [3] on $B(4, n)$.

2. **Burnside's bound for the order of $B(3, n)$.**

2.1. The group $B(3, 1)$ has order 3. Let $B_n = B(3, n)$ be finite with order θ_n. Then [2, p. 320] $B_{n+1} = \{B_n, a \mid x^3 = 1\}$; the elements of B_{n+1} are of the form $g = ua^{\pm 1}va^{\pm 1} \cdots a^{\pm 1}w$ where $u, v, w \in B_n$; using the relation $(yz)^3 = 1$ in the form $yzy = z^{-1}y^{-1}z^{-1}$ the number of terms $a^{\pm 1}$ in g can be reduced to at most two; all elements of B_{n+1} are comprised in the following set

$$u, \tag{1}$$

$$ua^{\pm 1}v, \tag{2}$$

$$uava^{-1}w, \tag{3}$$

where u, v, w range over B_n; this gives the estimate

$$\theta_n \leqq 3^{3^{n-1}}.$$

2.2. We now remark that in this set of "basic forms" (3) can be improved to

$$a^{-1}uav \tag{3'}$$

since

$$\begin{aligned} uava^{-1}w &= a^{-1}u^{-1}a^{-1}u^{-1}va^{-1}w = a^{-1}u_1(a^{-1}v_1a^{-1})w \\ &= a^{-1}u_1v_1^{-1}av_1^{-1}w = a^{-1}u_2av_2. \end{aligned}$$

Received by the editors November 30, 1959.

[1] From a doctoral thesis presented at the University of Manchester, 1954.

Hence

$$\theta_{n+1} \leqq \theta_n + 2\theta_n^2 + \theta_n(\theta_n - 1) = 3\theta_n^2.$$

Thus

$$3\theta_{n+1} \leqq (3\theta_n)^2 \leqq (3\theta_1)^{2^n},$$

and $\theta_1 = 3$, giving

$$\theta_n \leqq 3^{2^n - 1}.$$

This is the bound obtained by Burnside [1] by a more difficult method. It coincides with the actual order for $n = 1, 2, 3$, [2].

3. **A lower bound for the order of $B(p^r, n)$.**

3.1. The Frattini subgroup $\phi(G)$ of a finite p-group G may be defined as the join of the commutator subgroup $D(G)$ and the subgroup G^p generated by the pth powers of the elements of G. The order of $G/\phi(G)$ is p^n where n is the (invariant) number of elements in a set of independent generators of G [2].

The ϕ-series of G is defined inductively:

$$\phi_0(G) = G, \qquad \phi_{i+1}(G) = \phi(\phi_i(G)), \qquad i = 0, 1, 2, \cdots.$$

Let G be a finite p-group on n generators, such that $\phi_r(G) = \{1\}$. Since $g^{p^i} \in \phi_i(G)$ for every $g \in G$, this implies that $g^{p^r} = 1$. G is a homomorphic image $\alpha(F)$ of the free group F on n free generators. Let $\Psi(F)$ be the subgroup

$$\Psi(F) = \{D(F), F^p\}.$$

Define $\Psi_i(F)$ inductively:

$$\Psi_0(F) = F, \qquad \Psi_{i+1}(F) = \Psi(\Psi_i(F)), \qquad i = 0, 1, 2, \cdots.$$

Then

$$\begin{aligned}\phi(G) &= \{D(\alpha(F)), \alpha(F)^p\} = \{\alpha(D(F)), \alpha(F^p)\} \\ &= \alpha\{D(F), F^p\} = \alpha(\Psi(F)).\end{aligned}$$

Hence by induction

$$\phi_i(G) = \alpha(\Psi_i(F)), \qquad i = 0, 1, 2, \cdots.$$

Let N be the kernel of α. Then $\phi_r(G) = \{1\}$ if and only if $\Psi_r(F) \subseteq N$. Thus the order of G will be a maximum when

$$G \simeq F/\Psi_r(F).$$

3.2. The order of the elementary abelian group $F/\Psi(F)$ is p^n, thus by Schreier's theorem [2] $\Psi(F)$ is a free group on $n_1 = 1 + (n-1)p^n$ free generators. Let $\Psi_i(F)$ have n_i free generators. Then

$$n_0 = n, \qquad n_{i+1} = 1 + (n_i - 1)p^{n_i}, \qquad i = 0, 1, 2, \cdots,$$

and we have the following lemma:

LEMMA. *The greatest finite p-group G on n generators, such that $\phi_r(G) = \{1\}$, is the group*

$$G = F/\Psi_r(F)$$

and the order of this group is

$$p^{n+n_1+n_2+\cdots+n_{r-1}}$$

where

$$n_i = 1 + (n-1)p^{n+n_1+\cdots+n_{i-1}}, \qquad i = 0, 1, 2, \cdots.$$

This is the lower bound referred to, for the order of $B(p^r, n)$. When $p = r = 2$ it becomes

$$2^{n+1+(n-1)2^n}$$

which is the lower bound given by Sanov [3] for the order of $B(4, n)$.

This bound is not, of course, a good one—although it becomes less trivial, for fixed p, as r increases. Thus, it yields 2^7 as a lower bound for $B(4, 2)$ which is known to have order 2^{12}, and 3^n for $B(3, n)$ which has order

$$3^{n+{}^nC_1+{}^nC_2};$$

and, of course, the order of $B(p^r, n)$ may well be infinite. However, its main interest lies in the directness of its derivation.

REFERENCES

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3. I. N. Sanov, *On Burnside's problem*, Dokl. Akad. Nauk. SSSR. (N.S.) vol. 57 (1947) pp. 759–761 (Russian).

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