A GENERALIZATION OF A THEOREM OF N. ITÔ ON p-GROUPS[1]

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We shall use the following notation: G is a finite p-group; $\phi(G)$ is the Frattini subgroup of G; if M and N are subgroups of G, then (M, N) is the subgroup of G generated by the set of all expressions $m^{-1}n^{-1}mn$ for $m \in M$ and $n \in N$; $M \subset N$ means M is properly contained in N. The descending central series $G_1, G_2, \cdots, G_n$, of G is defined recursively by $G_1 = G$ and $G_{m+1} = (G, G_m)$.

N. Itô [2] has shown that if $H_2 \subset G_2$ then $(H\phi(G))_2 \subset G_2$. Our main result is the following generalization of Itô's theorem.

THEOREM 1. *If G is a finite p-group and H a subgroup of G such that $H_n \subset G_n$, then $(H\phi(G))_n \subset G_n$.*

We first prove two lemmas, the second of which is of independent interest.

LEMMA 1. *Let H be a subgroup of the p-group G. Then (G, H_n) is contained in any normal subgroup of G which contains (G_n, H).*

PROOF. The lemma is certainly true if $n = 1$, so suppose it is true for $n = k$ and all pairs $\{G, H\}$. It is known [1, Theorem 10.3.5] that $(G, H_{k+1}) = (G, (H_k, H))$ is contained in any normal subgroup of G which contains $((G, H_k), H)$ and $((G, H), H_k)$. But $(G, H_k) \subseteq G_{k+1}$, and by the induction assumption $((G, H), H_k)$ is contained in any normal subgroup of (G, H) which contains $((G, H)_k, H)$; hence $((G, H), H_k)$ is contained in any normal subgroup of G which contains $((G, H)_k, H)$. Since $(G, H)_k \subseteq G_{2k} \subseteq G_{k+1}$, it follows that $((G, H)_k, H)$ is contained in any normal subgroup of G which contains (G_{k+1}, H). Thus $(G, (H_k, H))$ is contained in any normal subgroup of G which contains (G_{k+1}, H), and the lemma follows by induction.

LEMMA 2. *If G is a finite p-group and H a subgroup of G such that $H_n \subset G_n$, then $H_n G_{n+1} \subset G_n$.*

PROOF. Let G be a group of minimal order such that there exist a subgroup K of G and an integer m for which the lemma is false, i.e., such that $K_m \subset G_m$ but $K_m G_{m+1} = G_m$. In G, let H be a subgroup of

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maximal order for which the lemma is false, and suppose that n is an integer such that $H_n \subset G_n$ but $H_n G_{n+1} = G_n$.

Certainly G_{n+1} contains a normal subgroup N of G such that $H_n N = G_n$, and if $M \subset N$ and M is normal in G, then $H_n M \subset G_n$. If N does not have order p there exists a subgroup $M \subset N$ such that $[N: M] = p$ and M is normal in G. Then $(HM/M)_n = H_n M/M \subset G_n/M = (G/M)_n$, while $(HM/M)_n (G/M)_{n+1} = (H_n G_{n+1})/M = G_n/M = (G/M)_n$. Thus the lemma is false for G/M; hence $M = \langle 1 \rangle$ since G is a group of minimal order for which the lemma is false. It follows that the normal subgroup N must have order p; hence N is contained in the center of G. Since $H_n \subset G_n$ and $H_n N = G_n$, we have $G_n = H_n \times N$.

Let K be the normalizer of H in G. If $K = G$, then H_n, as a characteristic subgroup of the normal subgroup H, is normal in G. But then $(G/H_n)_{n+1} = H_n G_{n+1}/H_n = G_n/H_n = (G/H_n)_n$. Since G/H_n is nilpotent, it follows that $(G/H_n) = \langle 1 \rangle$, but then $G_n = H_n$, which is a contradiction. Thus $K \subset G$.

Now $H_n G_{n+1} = G_n$ and $H \subset K$ imply that $K_n G_{n+1} = G_n$, therefore $K_n = G_n$ since otherwise the lemma is false for K, a larger subgroup than H. But also $K_{n+1} \subset G_{n+1}$, since if $K_{n+1} = G_{n+1}$, then $H_n K_{n+1} = G_n = K_n$ while $H_n \subset G_n = K_n$, and the lemma is false for K, a group of smaller order than G. Moreover, $K_{n+1} G_{n+2} \subset G_{n+1}$, since otherwise K would be a subgroup larger than H for which the lemma is false (in this case for $m = n+1$).

Clearly $H_{n+1} G_{n+2}$ is normal in G. Also, $(G_n, H) = (H_n \times N, H) = H_{n+1}$ since N is in the center of G, hence $(G_n, H) \subseteq H_{n+1} G_{n+2}$. Thus, by Lemma 1, $(G, H_n) \subseteq H_{n+1} G_{n+2}$. But $G_{n+1} = (G_n, G) = (H_n \times N, G) = (H_n, G)$, so $G_{n+1} \subseteq H_{n+1} G_{n+2} \subseteq K_{n+1} G_{n+2} \subset G_{n+1}$. This is a contradiction, and the lemma follows.

Proof of Theorem 1. We assume G is of minimal order and H is a subgroup of G having maximal order such that the theorem is false. If K is a subgroup of G such that $G \supset K \supset H$, then $K_n = G_n$. For if $K_n \subset G_n$, it follows from the maximality of H that $(K\phi(G))_n \subset G_n$, and consequently $(H\phi(G))_n \subset G_n$. Moreover, $(H\phi(K))_n \subset K_n$. For if $(H\phi(K))_n = K_n$, then since $H_n \subset G_n = K_n$, K is a smaller group than G for which the theorem is false. Thus $H\phi(K) \subseteq H$, and hence $\phi(K) \subseteq H$. Now also $(HG_2)_n \subseteq H_n G_{n+1}$, so that by Lemma 2 $(HG_2)_n \subset G_n$. Hence, $HG_2 \subseteq H$ and H is normal in G.

If $\langle x, H \rangle \subset G$ for every x in G, then $x^p \in \phi(\langle x, H \rangle) \subseteq H$ for every x in G. Thus in this case $P(G) \subseteq H$. But $G_2 \subseteq H$, so that $\phi(G) \subseteq H$ which is impossible.

Thus there is an x such that $\langle x, H \rangle = G$. Since $x^p \in \phi(G)$, $\langle x^p, H \rangle \subset G$. But then $x^{p^2} \in \phi(\langle x^p, H \rangle) \subseteq H$, so that H has index at most p^2 in G.

Since G is noncyclic, $\phi(G)$ is the intersection of all normal subgroups of index p^2 in G; hence $\phi(G) \subseteq H$ which is impossible.

The theorem is thus proved.

COROLLARY. *If G is a finite p-group which can be generated by two elements, and if $G_n \neq \langle 1 \rangle$, then $H_n \subset G_n$ for every proper subgroup H of G.*

PROOF. If $H \subset G$ there exists a subgroup N of G such that $[G: N] = p$ and $H \subseteq N$. Then $\phi(G) \subseteq N$, and since G can be generated by two elements, there exists an element x such that $N = \langle x, \phi(G) \rangle$. The corollary is trivial if $n = 1$. But if $n \geqq 2$ then $\langle x \rangle_n = 1$, so that by Theorem 1, $N_n = (\langle x \rangle \phi(G))_n \subset G_n$. The corollary follows, since $H_n \subseteq N_n$.

As another application of Lemmas 1 and 2 we prove the following.

THEOREM 2. *If G is a finite, nilpotent group and H is a subgroup of G such that $H_n = G_n$, then $H_{n+k} = G_{n+k}$ for every positive integer k.*

PROOF. It will suffice to prove the theorem for G a finite p-group. We prove that $H_{n+1} = G_{n+1}$. Note that since $(HG_2)_{n+1} \subseteq H_{n+1} G_{n+2}$, it would follow from $H_{n+1} \subset G_{n+1}$ and Lemma 2 that $(HG_2)_{n+1} \subset G_{n+1}$. Thus if the theorem fails it must fail for a normal subgroup H. But using Lemma 1 we have

$$G_{n+1} = (G, G_n) = (G, H_n) \subseteq (G_n, H) = (H_n, H) = H_{n+1}$$

if H is normal. Thus the theorem is proved.

REFERENCES

1. M. Hall, *The theory of groups*, New York, The Macmillan Company, 1959.

2. N. Itô, *On a theorem of L. Rédei and J. Szép concerning p-groups*, Acta. Univ. Szeged. vol. 14 (1952) pp. 186–187.

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