A GENERALIZATION OF A THEOREM OF N. ITÔ ON $\mathbf{p}$-GROUPS

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We shall use the following notation: $G$ is a finite $p$-group; $\phi(G)$ is the Frattini subgroup of $G$; if $M$ and $N$ are subgroups of $G$, then $(M, N)$ is the subgroup of $G$ generated by the set of all expressions $m^{-1}n^{-1}mn$ for $m \in M$ and $n \in N$; $M \subseteq N$ means $M$ is properly contained in $N$. The descending central series $G_1, G_2, \ldots, G_n$, of $G$ is defined recursively by $G_1 = G$ and $G_{m+1} = (G_m)$.

N. Itô [2] has shown that if $H_2 \subseteq G_2$ then $(H \phi(G))_2 \subseteq G_2$. Our main result is the following generalization of Itô's theorem.

**Theorem 1.** If $G$ is a finite $p$-group and $H$ a subgroup of $G$ such that $H_n \subseteq G_n$, then $(H \phi(G))_n \subseteq G_n$.

We first prove two lemmas, the second of which is of independent interest.

**Lemma 1.** Let $H$ be a subgroup of the $p$-group $G$. Then $(G, H_n)$ is contained in any normal subgroup of $G$ which contains $(G_n, H)$.

**Proof.** The lemma is certainly true if $n = 1$, so suppose it is true for $n = k$ and all pairs $\{G, H\}$. It is known [1, Theorem 10.3.5] that $(G, H_{k+1}) = (G, (H_k, H))$ is contained in any normal subgroup of $G$ which contains $((G, H_k), H)$ and $((G, H), H_k)$. But $(G, H_k) \subseteq G_{k+1}$, and by the induction assumption $((G, H_k), H)$ is contained in any normal subgroup of $(G, H)$ which contains $((G, H), H_k)$; hence $((G, H_k), H_k)$ is contained in any normal subgroup of $G$ which contains $((G, H), H)$. Since $(G, H)_k \subseteq G_k \subseteq G_{k+1}$, it follows that $((G, H)_k, H)$ is contained in any normal subgroup of $G$ which contains $(G_{k+1}, H)$. Thus $(G, (H_k, H))$ is contained in any normal subgroup of $G$ which contains $(G_{k+1}, H)$, and the lemma follows by induction.

**Lemma 2.** If $G$ is a finite $p$-group and $H$ a subgroup of $G$ such that $H_n \subseteq G_n$, then $H_n G_{n+1} \subseteq G_n$.

**Proof.** Let $G$ be a group of minimal order such that there exist a subgroup $K$ of $G$ and an integer $m$ for which the lemma is false, i.e., such that $K_m \subseteq G_m$ but $K_m G_{m+1} = G_m$. In $G$, let $H$ be a subgroup of $\ldots$
maximal order for which the lemma is false, and suppose that \( n \) is an integer such that \( H_n \subseteq G_n \) but \( H_n G_{n+1} = G_n \).

Certainly \( G_{n+1} \) contains a normal subgroup \( N \) of \( G \) such that \( H_n N = G_n \), and if \( M \subseteq N \) and \( M \) is normal in \( G \), then \( H_n M \subseteq G_n \). If \( N \) does not have order \( p \) there exists a subgroup \( M \subseteq N \) such that \( [N : M] = p \) and \( M \) is normal in \( G \). Then \( (HM/M)_n = H_n M/M \subseteq G_n / M = (G/M)_n \), while \( (HM/M)_n (G/M)_{n+1} = (H_n G_{n+1}) / M = G_n / M \) = \( (G/M)_n \). Thus the lemma is false for \( G/M \); hence \( M = \langle 1 \rangle \) since \( G \) is a group of minimal order for which the lemma is false. It follows that the normal subgroup \( N \) must have order \( p \); hence \( N \) is contained in the center of \( G \). Since \( H_n \subseteq G_n \) and \( H_n N = G_n \), we have \( G_n = H_n \times N \).

Let \( K \) be the normalizer of \( H \) in \( G \). If \( K = G \), then \( H_n \), as a characteristic subgroup of the normal subgroup \( H \), is normal in \( G \). But then \( (G/H_n)_{n+1} = H_n G_{n+1} / H_n = G_n / H_n = (G/H_n)_n \). Since \( G/H_n \) is nilpotent, it follows that \( (G/H_n) = \langle 1 \rangle \), but then \( G_n = H_n \), which is a contradiction. Thus \( K \subseteq G \).

Now \( H_n G_{n+1} = G_n \) and \( H \subseteq K \) imply that \( K_n G_{n+1} = G_n \), therefore \( K_n = G_n \) since otherwise the lemma is false for \( K \), a larger subgroup than \( H \). But also \( K_{n+1} \subseteq G_{n+1} \), since if \( K_{n+1} = G_{n+1} \), then \( H_n K_{n+1} = G_n = K_n \) while \( H_n \subseteq G_n = K_n \), and the lemma is false for \( K \), a group of smaller order than \( G \). Moreover, \( K_{n+1} G_{n+2} \subseteq G_{n+1} \), since otherwise \( K \) would be a subgroup larger than \( H \) for which the lemma is false (in this case for \( m = n + 1 \)).

Clearly \( H_{n+1} G_{n+2} \) is normal in \( G \). Also, \( (G_n, H) = (H_n \times N, H) = H_{n+1} \) since \( N \) is in the center of \( G \), hence \( (G_n, H) \subseteq H_{n+1} G_{n+2} \). Thus, by Lemma 1, \( (G_n, H_n) \subseteq H_{n+1} G_{n+2} \). But \( G_{n+1} = (G_n, G) = (H_n \times N, G) = (H_n, G) \), so \( G_{n+1} \subseteq H_{n+1} G_{n+2} \subseteq K_{n+1} G_{n+2} \subseteq G_{n+1} \). This is a contradiction, and the lemma follows.

**Proof of Theorem 1.** We assume \( G \) is of minimal order and \( H \) is a subgroup of \( G \) having maximal order such that the theorem is false. If \( K \) is a subgroup of \( G \) such that \( G \supseteq K \supseteq H \), then \( K_n = G_n \). For if \( K_n \subseteq G_n \), it follows from the maximality of \( H \) that \( (K \Phi(G))_n \subseteq G_n \), and consequently \( (K \Phi(G))_n \subseteq G_n \). Moreover, \( (K \Phi(K))_n \subseteq K_n \). For if \( (K \Phi(K))_n = K_n \), then since \( H_n \subseteq G_n = K_n \), \( K \) is a smaller group than \( G \) for which the theorem is false. Thus \( H \Phi(K) = H \), and hence \( \phi(K) \subseteq H \). Now also \( (H_G)_n \subseteq H_G \), so that by Lemma 2 \( (H_G)_n G_n \subseteq G_n \). Hence, \( H_G \subseteq H \) and \( H \) is normal in \( G \).

If \( \langle x, H \rangle \subseteq G \) for every \( x \) in \( G \), then \( x^p \in \Phi(\langle x, H \rangle) \subseteq H \) for every \( x \) in \( G \). Thus in this case \( P(G) \subseteq H \). But \( G \subseteq H \), so that \( \Phi(G) \subseteq H \) which is impossible.

Thus there is an \( x \) such that \( \langle x, H \rangle = G \). Since \( x^p \in \Phi(\langle x, H \rangle) \subseteq G \). But then \( x^p \in \Phi(\langle x^p, H \rangle) \subseteq H \), so that \( H \) has index at most \( p^2 \) in \( G \).
Since $G$ is noncyclic, $\phi(G)$ is the intersection of all normal subgroups of index $p^2$ in $G$; hence $\phi(G) \subseteq H$ which is impossible.

The theorem is thus proved.

**Corollary.** If $G$ is a finite $p$-group which can be generated by two elements, and if $G_n \neq \langle 1 \rangle$, then $H_n \subseteq G_n$ for every proper subgroup $H$ of $G$.

**Proof.** If $H \subseteq G$ there exists a subgroup $N$ of $G$ such that $[G: N] = p$ and $H \subseteq N$. Then $\phi(G) \subseteq N$, and since $G$ can be generated by two elements, there exists an element $x$ such that $N = \langle x, \phi(G) \rangle$. The corollary is trivial if $n = 1$. But if $n \geq 2$ then $\langle x \rangle_n = 1$, so that by Theorem 1, $N_n = \langle \langle x \rangle \phi(G) \rangle_n \subseteq G_n$. The corollary follows, since $H_n \subseteq N_n$.

As another application of Lemmas 1 and 2 we prove the following.

**Theorem 2.** If $G$ is a finite, nilpotent group and $H$ is a subgroup of $G$ such that $H_n = G_n$, then $H_{n+k} = G_{n+k}$ for every positive integer $k$.

**Proof.** It will suffice to prove the theorem for $G$ a finite $p$-group. We prove that $H_{n+1} = G_{n+1}$. Note that since $(HG_2)_{n+1} \subseteq H_{n+1}G_{n+2}$, it would follow from $H_{n+1} \subseteq G_{n+1}$ and Lemma 2 that $(HG_2)_{n+1} \subseteq G_{n+1}$. Thus if the theorem fails it must fail for a normal subgroup $H$. But using Lemma 1 we have

$$G_{n+1} = (G, G_n) = (G, H_n) \subseteq (G_n, H) = (H_n, H) = H_{n+1}$$

if $H$ is normal. Thus the theorem is proved.

**References**


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