A TOTALLY ORDERED INTEGRAL DOMAIN WITH A
CONVEX LEFT IDEAL WHICH IS NOT AN IDEAL

CHARLES HOLLAND

The question of the existence of such a ring was raised by Donald
Johnson.1 It suffices to find a cancellative totally ordered semigroup
which has a lower segment I that is a left ideal but not an ideal. For
suppose that S is such a semigroup and let R be the semigroup ring
of S over the integers Z. Then R can be ordered lexicographically
(first by S and then by Z) and it is well known that R is an ordered
integral domain. Let $\mathcal{S}$ be the set of all elements of R of the form
$c_1x_1 + \cdots + c_n x_n$, where $\gamma_1, \ldots, \gamma_n \in I$. Since I is a lower segment
of S, $\mathcal{S}$ is a convex subgroup of R and it is easy to verify that $\mathcal{S}$ is a
left ideal but not an ideal.

We now construct such a semigroup. The following ordered group,
from which we shall extract S, is due to Conrad [1]. Let $H$ be the
small direct sum of the integers over the integers ordered lexicographically. Define $\sigma$ from the integers to the ordered automorphism
group of $H$ in the following manner: for $a \in H$ we use $a_j$ to denote
the jth component of $a$; then let $(a_\sigma (i))_j = a_{j-i}$. Now let $G = \mathbb{Z} \times H$ ordered
lexicographically first by $\mathbb{Z}$ then by $H$, where $(i, a) + (j, b) = (i+j, a\sigma (i)+b)$. Then $G$ is an ordered group. Next let
$$S = \{(i, a) \in G \mid i \leq 0, a_j \geq 0 \text{ for all } j, \text{ and } a_j = 0 \text{ for } j > 0\}.$$ 
It is easily seen that $S$ is a subsemigroup of $G$. Clearly $S$ is cancellative and contains the identity $(0, 0)$ of $G$. Let
$$I = \{(i, a) \in S \mid \text{either } i = -1 \text{ and } a_0 = 0, \text{ or } i < -1\}.$$ Then I is a lower segment of S which is a left ideal. But I is not a right
ideal because $(-1, 0) \in I$ and if $(0, b) \in S$ such that $b_0 > 0$ then $(-1, 0)
+ (0, b) = (-1, b) \notin I$.

Reference

pp. 516–528.

Tulane University

Received by the editors December 5, 1959.

1 After this paper was submitted for publication the author learned that Johnson
found a different solution to the problem, which will be published.