SOME RESULTS ON TAME DISKS AND SPHERES IN $E^3$

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It is well known that there are cells and spheres of dimensions one and two in Euclidean 3-space $E^3$ which fail to be locally tame at a single point. Artin and Fox [1] have given many examples. Several authors have given characterizations, such as [11; 12 and 13], of the tame cells among the “almost polyhedral” cells. The results in this paper are of the same general nature.

We use the term “disk” to mean a topological closed 2-cell. If $D$ is a disk, then $D^0$ denotes the interior of $D$ and $\partial D$ denotes the boundary of $D$. The same notation also will be applied to an arc (a topological closed 1-cell).

The Theorem 0 below is stated without proof here, since a complete proof appears in Lemma 5.1 of [7]. We have used this result extensively and have noticed many modifications of it in the literature. It therefore seems useful to state it explicitly.

**Theorem 0.** Let $D$ be a disk in $E^3$ which is locally polyhedral except at points of $\partial D$. If $U$ is any open set containing $D^0$, then there is a 3-cell $C^3$ in $U \cup \partial D$ such that (i) $D \subseteq C^3$, (ii) $D$ spans $\partial D$, (iii) and $C^3$ is locally polyhedral except on $\partial D$.

**Theorem 1.** Let $D$ be a disk in $E^3$ and let $A$ be a tame arc on $D$ spanning $\partial D$. If $D$ is locally polyhedral except on $A$, then $D$ is tame.

**Proof.** A remark will suffice to show that $D$ is locally tame at each point of $D^0$. In [4] it is shown that each point $x$ in $A^0$ lies in the

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interior of a disk $P_x$ in $D^0$ and that $P_x$ is the union of two tame disks meeting in an arc on the boundary of each. Then, by Theorem 1 of [5], $P_x$ is tame and hence $D$ is locally tame at $x$.

In view of Theorem 2 of [10] we may assume that $D$ is locally polyhedral except on a straight line interval $A$. At an endpoint $p$ of $A$ we attach an interval $B$ in the same line as $A$. There is a subdisk $D'$ of $D$ such that $A$ spans $BdD'$ and $D' \cap B = p$. Furthermore, $D'$ is locally tame except perhaps on $BdA$. By Theorem 0 and [3] there is a 2-sphere $S$ which is locally tame except on $BdA$ such that $S \cap D = BdD'$, $S \cap B = p$ and the components of $(A \cup B) - p$ lie in different components of $E^3 - S$. But then $S$ is pierced at the point $p$ by the interval $A \cup B$. Hence, by Theorem 1 of [11], $S$ is locally tame at $p$. Now on the sphere $S$ we may select a tame arc $A_1$ such that $A_1 \cap D = p$. Let $U$ be an open set in $E^3$ such that $U$ contains $D^0$ but fails to meet $A_1 \cup BdD$. In view of Theorem 0 and [3], there is a sphere $S_1$ in $U \cup BdD$ such that $D^0$ lies in the bounded component of $E^3 - S_1$ and $S_1$ is locally tame except on $BdA$. Since the tame arc $A \cup A_1$ pierces $S_1$ at $p$, $S_1$ is locally tame at $p$. It then follows that $BdD$ is locally tame at $p$. Thus $D$ is locally tame everywhere and is therefore tame by [2] and [12].

**Theorem 2.** Let $S$ be a topological 2-sphere in $E^3$ and let $A$ be a tame arc on $S$. If $S$ is locally polyhedral except on $A$, then $S$ is tame.

**Proof.** On $S$ we select a disk $D$ having $A$ as a spanning arc of $BdD$ while $BdD$ is locally polygonal except at $BdA$. By Theorem 1, $D$ is tame. Then by [12] or [13], $S - D^0$ is tame and Theorem 9.3 of [13] implies that $S$ is tame.

Combining Theorem 2 of [5] with Theorem 2 above, the following result is immediate.

**Corollary 1.** Let $S$ be a topological 2-sphere in $E^3$ and let $G$ be a tame graph (finite, connected 1-complex) on $S$. If $S$ is locally polyhedral (or is locally tame) except on $G$, then $S$ is tame.

**Theorem 3.** Let $D$ be a disk in $E^3$ and let $A$ be a tame arc on $D$. If $D$ is locally polyhedral except on $A$, then $D$ is tame.

**Proof.** That $D$ is locally tame at each point of $D^0$ can be established in the same manner as was used in Theorem 1. In view of Theorem 9 of [2] we can throw $D$ by a space homeomorphism $h$ onto a disk $D'$ which is locally polyhedral except on $BdD'$. Theorem 0 implies that $D'$ lies on a sphere $S$ which is locally polyhedral except on $BdD'$. Then $S$ is locally tame except perhaps on $h(A)$. But by Theorem 2, $S$ is tame and hence $D'$ and $D$ are tame.
In [5] a characterization of tame graphs is given. The above results permit another such characterization which we state without proof.

**Corollary 2.** Let $G$ be a graph in $E^3$ such that each 1-cell in $G$ is tame. If the star of every vertex of $G$ lies on a disk, then $G$ is tame.

The construction used in the proof of Theorem 1 has suggested the following definition: Let $S$ be a topological 2-sphere in $E^3$ and let $T$ be a tame arc in $E^3$ such that $S \cap T$ is an endpoint of $T$. Then $T$ is **unknotted relative to $S$** if $T$ lies on the boundary of a disk $D$ in $E^3$ such that $D \cap S$ is an arc in $BdD$. If no such disk exists, then $T$ is **knotted relative to $S$**. (This definition may be compared with similar properties used by Harrold [9].)

**Theorem 4.** Let $S$ be a topological 2-sphere in $E^3$ which is locally polyhedral except at a point $p$. If there is a tame arc $T$ in $(E^3 - S) \cup p$ with endpoint $p$ and if $T$ is unknotted relative to $S$, then $S$ is tame.

**Proof.** By assumption there is a disk $D$ containing $T$ in $BdD$ such that $S \cap D = S \cap BdD$ is an arc $A$. Using the Bing approximation theorem [3] replace $D$ by a disk $D'$ which is locally polyhedral except on $A \cup T$. Then since $S$ is locally polyhedral except at $p$, we may take $A$ to be locally polygonal except at $p$. Then $D'$ is tame by Theorem 3 and in particular the arc $A$ is tame. So $S$ is tame by Theorem 2.

This last result is of interest when considering such examples as 3.1 of [1]. Also in this connection, we note that any sphere in $E^3$ is accessible by intervals at a dense subset from each of its complementary domains.

The following result seems to have escaped notice in previous discussions of almost polyhedral spheres.

**Theorem 5.** Let $S$ be a 2-sphere in $E^3$ which is locally polyhedral except at a point $p$ and let $A_1$ and $A_2$ be two arcs on $S$. If $A_1 \cap A_2 = p$ and $p$ is an endpoint of both arcs, then $A_1$ and $A_2$ are equivalently imbedded in $E^3$; that is, there is a space homeomorphism carrying $A_1$ onto $A_2$.

**Proof.** Clearly $A_1$ and $A_2$ lie on the boundary of a disk $D$ in $S$. We join the distinct endpoints of $A_1$ and $A_2$ by a polygonal arc $J_0$, disjoint from $A_1$ and $A_2$ except at the endpoints, to form the disk $D$. On $D$ we select a sequence of disjoint polygonal arcs $\{J_i\}$ converging to the point $p$ and such that each $J_i$ has its endpoints on $A_1$ and $A_2$. We enclose each $J_i$, $i = 0, 1, 2, \ldots$, in a polyhedral 2-sphere $S_i$ meeting $S$ in a simple closed curve which contains $J_i$ in its interior.
The spheres $S_i$ are taken to be pairwise disjoint. Then there is a space homeomorphism $h_i$ which is the identity except inside $S_i$ and carries the endpoint of $J_i$ on $A_i$ onto the endpoint of $J_i$ on $A_2$ while leaving the sphere $S$ invariant. This homeomorphism $h_i$ is so chosen that $h_i(A_1) \cap A_2$ is a single point. Applying these mappings $h_i$ sequentially, the limit mapping $h$ is obviously a space homeomorphism such that $h(A_1) \cap A_2$ is a sequence of points $\{p_i\}$ converging to $p$. This gives us a sequence of 2-cells $D_i$ on $S$ bounded by $A_2$ and $h(A_1)$ such that $D_i \cap D_j$ is empty if $j \neq i-1, i, i+1$ and otherwise $D_i \cap D_{i+1} = p_{i+1}$.

Since these cells $D_i$ are all tame we can construct 2-spheres $S_i'$ such that $D_i - p_i - p_{i+1}$ lies in the bounded component of $E^3 - S_i'$ and the $S_i'$ intersect as do the $D_i$. Then space homeomorphisms $h_i'$ which are the identity except inside $S_i'$ will move $h(A_1)$ entirely onto $A_2$ while leaving $S$ invariant and the point $p$ fixed. Therefore $A_1$ and $A_2$ are equivalently imbedded in $E^3$.

**Corollary 3.** If $A_1$ and $A_2$ are arcs passing through $p$ in the 2-sphere $S$ of Theorem 5 and if $A_1 \cap A_2 = p$, then $A_1$ and $A_2$ are equivalently imbedded in $E^3$.

In [8] Harrold and Moise show that each 2-sphere $S$ in $S^3$ such that $S$ is locally polyhedral except at a point $p$ has a closed 3-cell as the closure of one of its complementary domains. It is clear from the above observations that any almost polyhedral spanning disk in this 3-cell which has the point $p$ on its boundary will contain arcs imbedded equivalently to those on $S$ itself. But in addition we have the following result.

**Corollary 4.** Every 2-sphere in $E^3$, locally polyhedral except at one point, is accessible from both of its complementary domains by arcs imbedded in $E^3$ equivalently to arcs on the sphere.

We conclude by collecting several results which are very easily established but which seem to be of interest.

**Theorem 6.** Let $D$ be a disk ($S$ be a sphere) in $E^3$ which is locally polyhedral except at a point $p$ in $D^0$ (at a point $p$) and suppose all arcs on $D(S)$ are tame. Then $D(S)$ is tame.

**Proof.** By [6], $D(S)$ is pierced by a tame arc at $p$. Then Theorem 1 of [11] may be applied to prove that $D(S)$ is tame.

**Corollary 5.** If $S$ is a 2-sphere in $E^3$ on which every arc is tame and if $S$ is locally polyhedral except on a closed, totally disconnected set $M$, then $S$ is tame.
Proof. Pass an arc $A$ on $S$ through $M$. $A$ is tame and Theorem 2 applies.

Corollary 6. Let $D$ be a disk ($S$ be a sphere) in $E^3$ which is locally polyhedral except at a point $p$ of $D^0$ (of $S$) and suppose $D(S)$ is wild at $p$. Then every arc through $p$ is wild.

Proof. That some such arc through $p$ is wild follows from Theorem 6 and then the conclusion follows from Theorem 5.

We remark that Corollary 6 may be extended in the obvious way to a sphere which is wild at finitely many points. Furthermore, it follows that a disk cannot be locally wild at an isolated point without containing a wild arc. Stating this conversely, a disk on which all arcs are tame cannot be locally wild at an isolated point (Corollary 6) and hence cannot be wild only on a closed totally disconnected set of points (Corollary 5).

This last remark is of interest in view of the recent announcement by R. H. Bing of a wild sphere on which all arcs are tame.

References


11. ———, Affine structures in 3-manifolds VII. Disks which are pierced by intervals, Ann. of Math. vol. 58 (1953) pp. 403–408.
