TOPOLOGICAL AND MEASURE THEORETIC PROPERTIES OF ANALYTIC SETS

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1. Introduction. G. Choquet's definition [1] of an analytic set in a topological space as the continuous image of a $K_{c\in}$ has proved to be a very fruitful one. In this paper we study some properties of analytic sets which are not necessarily included in a $K_{c\in}$. In particular, we prove that an analytic set is Lindelöf, capacitable, measurable and can be approximated in measure by compact sets from within for large classes of capacities and measures.

2. Topological properties of analytic sets.

2.1. Definitions. Our definition of an analytic set below is a slight modification of Choquet's in that we do not impose any conditions on $X$.

1. $\omega$ denotes the set of all non-negative integers.
2. $K(X)$ denotes the family of compact sets in $X$.
3. $\mathcal{F}(X)$ denotes the family of closed sets in $X$.
4. $A$ is analytic iff $A$ is the continuous image of a $K_{c\in}(X)$ for some $X$.
5. $A$ is Souslin $H$ iff

$$A = \bigcup_{s \in S} \bigcap_{n \in \omega} E(s_0, \ldots, s_n)$$

where $S$ is the set of all sequences of non-negative integers and $E(s_0, \ldots, s_n) \in \mathcal{H}$ for every $s \in S$ and $n \in \omega$.
6. $A$ is Lindelöf iff every open covering of $A$ has a countable subcovering of $A$.
7. $\overline{A}$ denotes the closure of $A$.

2.2. Lemma. Let $f$ be a continuous function on $D$ to a topological space $X$ and, for every $n \in \omega$, let $A_n$ be compact, $A_{n+1} \subseteq A_n$, $\cap_{n \in \omega} A_n \subseteq D$, $B_n = f(D \cap A_n)$. Then

(i) if $U$ is open in $X$ and $f(\cap_{n \in \omega} A_n) \subseteq U$ then, for some $n \in \omega$, $B_n \subseteq U$,
(ii) if $X$ is Hausdorff then

$$f\left(\bigcap_{n \in \omega} A_n\right) = \bigcap_{n \in \omega} B_n = \bigcap_{n \in \omega} \overline{B}_n.$$
Proof. Let $C' = \cap_{n \in \omega} A_n$ and $C = f(C')$. Suppose $U$ is open in $X$ and $C \subseteq U$. Let $U' = f^{-1}(U)$. Then $U'$ is open in $D$ and $C \subseteq U'$. Hence, for some $n \in \omega$, $D \cap A_n \subseteq U'$ and

$$B_n = f(D \cap A_n) \subseteq f(U') = U.$$

Next, suppose $X$ is Hausdorff and $y \in C$. Since $C'$ is compact so is $C$. Thus, there exists an open set $U$ such that $C \subseteq U$ and $y \in \overline{U}$. Then, by part (i), for some $n \in \omega$, $B_n \subseteq \overline{U}$ and $y \notin \overline{B_n}$. Thus,

$$\cap_{n \in \omega} \overline{B_n} \subseteq C.$$

On the other hand, we always have

$$C \subseteq \cap_{n \in \omega} B_n \subseteq \cap_{n \in \omega} \overline{B_n}.$$

2.3. Theorem. If $E$ is analytic then $E$ is Lindelöf.

Proof. Since $E$ is the continuous image of a $K_{\sigma}(X)$, for some $X$, let

$$D = \cap_{i \in \omega} \cup_{j \in \omega} d(i, j),$$

where the $d(i, j)$ are compact, $f$ be continuous on $D$ and $E = f(D)$. Suppose $F$ is an open covering of $E$ that has no countable subcovering of $E$. Then, by recursion, we define a sequence $k$ with $k_i \in \omega$ such that for any $n \in \omega$, no countable subfamily of $F$ covers

$$f\left(D \cap \cap_{i=0}^n d(i, k_i)\right).$$

To this end, we observe that, since

$$E = \cup_{j \in \omega} f(D \cap d(0, j)),$$

there exists $k_0 \in \omega$ such that no countable subfamily of $F$ covers $f(D \cap d(0, k_0))$. Having defined $k_i$ for $i = 0, \cdots, n$, since

$$f\left(D \cap \cap_{i=0}^n d(i, k_i)\right) = \cup_{j \in \omega} f\left(D \cap \cap_{i=0}^n d(i, k_i) \cap d(n + 1, j)\right)$$

we see that there exists $k_{n+1} \in \omega$ such that no countable subfamily of $F$ covers

$$f\left(D \cap \cap_{i=0}^{n+1} d(i, k_i)\right).$$

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Let
\[ A_n = \bigcap_{i=0}^{n} d(i, k_i), \]
\[ B_n = f(D \cap A_n), \]
\[ C = f\left( \bigcap_{n \in \omega} A_n \right). \]

Since
\[ \bigcap_{n \in \omega} A_n = \bigcap_{i \in \omega} d(i, k_i) \subseteq D \]
and \( C \) is compact, let \( U \) be the union of a finite subfamily of \( F \) such that \( C \subseteq U \). Then, by Lemma 2.2, for some \( n \in \omega \), \( B_n \subseteq U \) contradicting the fact that \( B_n \) could not be covered by a countable subfamily of \( F \).

2.4. COROLLARY. If \( E \) is analytic in a metric space then \( E \) is separable.

2.5. THEOREM. If \( E \) is analytic in a Hausdorff \( X \) then \( E \) is Souslin \( \mathcal{F}(X) \).

Proof. Let \( E = f(D) \) where \( f \) is continuous on \( D \),
\[ D = \bigcap_{i \in \omega} \bigcup_{j \in \omega} d(i, j), \]
with \( d(i, j) \) compact. For every \((n+1)\)-tuple \((k_0, \cdots, k_n)\) with \( k_i \in \omega \), let
\[ B(k_0, \cdots, k_n) = f\left( D \cap \bigcap_{i=0}^{n} d(i, k_i) \right) \]
so that \( B(k_0, \cdots, k_n) \in \mathcal{F}(X) \). If \( S \) denotes the set of all sequences \( k \) with \( k_i \in \omega \), we shall show that
\[ E = \bigcup_{k \in S} \bigcap_{n \in \omega} B(k_0, \cdots, k_n). \]
Indeed, since
\[ D = \bigcup_{k \in S} \bigcap_{i \in \omega} d(i, k_i) \]
we have
\[ E = \bigcup_{k \in S} f\left( \bigcap_{i \in \omega} d(i, k_i) \right). \]
and for every $k \in S$

$$f\left( \bigcap_{i \in \omega} \cap d(i, k_i) \right) = f\left( \bigcap_{n \in \omega} \bigcap_{i = 0}^n d(i, k_i) \right) = (\text{by 2.2})$$

$$= \bigcap_{n \in \omega} f\left( D \bigcap_{i = 0}^n d(i, k_i) \right) = \bigcap_{n \in \omega} B(k_0, \cdots, k_n).$$

2.6. Corollary. If $E$ is a metric space and analytic then $E$ can be imbedded in a complete separable metric space $X$ and, for every such $X$, $E$ is Souslin $\mathcal{S}(X)$. Thus, in this case, $E$ is an absolute analytic set in the classical sense and hence it is the continuous image of the set of irrationals. (See [3] for results on analytic sets in separable metric spaces.)

2.7. Remark. Theorem 2.5 was first proved by the author in [5]. The proof we give here is shorter and simpler. Corollary 2.6 was first obtained by G. Choquet [2].

3. Capacitability of analytic sets. We consider here only special kinds of capacities. For more general definitions and results about capacities see G. Choquet [1].

3.1. Definitions. 1. $\phi$ is a capacity on $X$ iff $\phi$ is a function on $K(X)$ to the reals (including $\pm \infty$) such that: if $A$ and $B$ are compact and $A \subset B$ then $\phi(A) \leq \phi(B)$; if $\phi(A) < a$ then there exists an open set $U$ such that $A \subset U$ and, for every compact $A' \subset U$, $\phi(A') < a$.

2. If $\phi$ is a capacity on $X$ then for any $A \subset X$ we let

$$\phi^*(A) = \sup\{t : t = \phi(C) \text{ for some compact } C \subset A\}$$

and

$$\phi^*(A) = \inf\{t : t = \phi^*(U) \text{ for some open } U \supset A\}.$$  

3. If $\phi$ is a capacity on $X$ then $A$ is $\phi$-capacitable iff

$$\phi^*(A) = \phi^*(A).$$

4. If $\phi$ is a capacity on $X$ and $E \subset X$ then $\phi$ is of order $(1a)$ on $E$ iff, for every sequence $A$ with $A_i \subset A_{i+1} \subset E$, we have

$$\phi^*\left( \bigcup_{i \in \omega} A_i \right) = \lim_{i \to \infty} \phi^*(A_i).$$

(This is a slight extension of Choquet's definition of a capacity alternating of order $\mathcal{A}_{1,a}$.)

3.2. Theorem. If $E$ is analytic in $X$, $\phi$ is a capacity on $X$ of order $(1a)$ on $E$ then $E$ is $\phi$-capacitable.
PROOF. Let $E = f(D)$ where $f$ is continuous on $D$,

$$D = \bigcap_{i \in \omega} \bigcup_{j \in \omega} d(i, j),$$

the $d(i, j)$ are compact, and $d(i, j) \subseteq d(i, j+1)$ for every $i \in \omega, j \in \omega$. In order to see that $\phi_* (E) = \phi^* (E)$, we let $a < \phi^* (E)$ and show that, for some compact $C \subseteq E$, $\phi(C) \geq a$. By recursion, we define a sequence $k$ with $k_i \in \omega$ as follows. Since

$$E = \bigcup_{j \in \omega} f(D \cap d(0, j))$$

let $k_0 \in \omega$ be such that

$$a < \phi^* (f(D \cap d(0, k_0))).$$

Having $k_0, \ldots, k_n$ so that

$$a < \phi^* \left( f \left( D \cap \bigcap_{i=0}^{n} d(i, k_i) \right) \right),$$

since

$$f \left( D \cap \bigcap_{i=0}^{n} d(i, k_i) \right) = \bigcup_{j \in \omega} f \left( D \cap \bigcap_{i=0}^{n} d(i, k_i) \cap d(n + 1, j) \right),$$

let $k_{n+1}$ be such that

$$a < \phi^* \left( D \cap \bigcap_{i=0}^{n+1} d(i, k_i) \right).$$

Set

$$A_n = \bigcap_{i=0}^{n} d(i, k_i),$$

$$C = f \left( \bigcap_{n \in \omega} A_n \right).$$

Then, since

$$\bigcap_{n \in \omega} A_n = \bigcap_{i \in \omega} d(i, k_i) \subseteq D,$$

we see that $C$ is compact, $C \subseteq E$ and, by 2.2, for any open set $U \supseteq C$ there exists $n \in \omega$ such that $f(D \cap A_n) \subseteq U$. Hence

$$a < \phi^* (D \cap A_n) \leq \phi^* (U).$$
Thus, $a \leq \phi^*C$ and, since compact sets are $\phi$-capacitable, we have $\phi(C) = \phi^*(C) \geq a$.

4. **Measurability of analytic sets.**

4.1. **Definitions.**
1. $\mu$ is a Carathéodory measure on $X$ iff $\mu$ is a function on the family of all the subsets of $X$ to the non-negative reals (including $\infty$) such that $\mu(0) = 0$ and if
   \[ A \subset \bigcup_{n \in \omega} B_n \subset X \]
   then
   \[ \mu(A) \leq \sum_{n \in \omega} \mu(B_n). \]

2. If $\mu$ is a Carathéodory measure on $X$ and $A \subset X$ then $A$ is $\mu$-measurable iff for every $T \subset X$
   \[ \mu(T) = \mu(T \cap A) + \mu(T - A). \]

3. $M(X, E)$ will denote the set of all Carathéodory measures on $X$ such that:
   (i) for every $A \subset E$ there is a $\mu$-measurable $A'$ with $A \subset A'$ and $\mu(A) = \mu(A')$.
   (ii) all compact subsets of $E$ are $\mu$-measurable.
   (iii) if $A$ is compact, $A \subset E$, $T \subset X$, $\varepsilon > 0$ then there exists an open set $U$ such that $A \subset U$ and $\mu(T \cap U) \leq \mu(T \cap A) + \varepsilon$.

4.2. **Theorem.** If $E$ is analytic in $X$, $\mu \in M(X, E)$, $a < \mu(E)$ then there exists a compact $C$ such that $C \subset E$ and $\mu(C) > a$.

**Proof.** For any sequence $A$ such that $A_i \subset A_{i+1} \subset E$ we have
\[
\mu \left( \bigcup_{i \in \omega} A_i \right) = \lim_{i} \mu(A_i).
\]

Let $E = f(D)$ where $f$ is continuous on $D$, 
\[ D = \bigcap_{i \in \omega} \bigcup_{j \in \omega} d(i, j), \]
the $d(i, j)$ are compact and $d(i, j) \subset d(i, j+1)$. As in the proof of 3.2, by recursion we can define a sequence $k$ with $k_i \in \omega$ such that, for any $n \in \omega$:
\[
a < \mu \left( f \left( D \cap \bigcap_{i=0}^{n} d(i, k_i) \right) \right).
\]
Let
\[ A_n = \bigcap_{i=0}^{n} d(i, k_i), \]
\[ C = f \left( \bigcap_{n \in \omega} A_n \right) \]
then \( C \) is compact, \( C \subseteq E \) and, by 2.2, for any open set \( U \supseteq C \) there exists \( n \in \omega \) such that \( f(D \cap A_n) \subseteq U \) and therefore
\[ a < \mu(f(D \cap A_n)) \leq \mu(U). \]
Thus, \( \mu(C) \geq a. \)

4.3. Theorem. If \( E \) is analytic in \( X \), \( \mu \in M(X, E) \) then \( E \) is \( \mu \)-measurable.

Proof. Let \( T \subseteq X \) and \( \mu(T) < \infty \). We define a \( n \in \omega \) measure \( \nu \) on \( X \) by letting \( \nu(A) = \mu(T \cap A) \) for every \( A \subseteq X \).

Clearly \( \nu \in M(X, E) \) and \( \nu(E) < \infty \). Then, by 4.2, there exists a \( C \subseteq E \) such that \( C \) is a \( K_\nu(X) \) and is \( \nu \)-measurable and \( \nu(C) = \nu(E) \), so that \( \nu(E - C) = 0 \). Thus, \( E - C \) is \( \nu \)-measurable and hence \( E \) is \( \nu \)-measurable and
\[ \mu(T) = \nu(T) = \nu(T \cap E) + \nu(T - E) = \mu(T \cap E) + \mu(T - E). \]

4.4. Corollary. If \( X \) is a complete separable metric space, \( E \) is closed in \( X \), and \( \mu \in M(X, E) \) then \( E \) is \( \mu \)-measurable.

4.5. Remark. If \( X \) is Hausdorff, \( E \) is analytic in \( X \), \( \mu \) is a Carathéodory measure on \( X \) such that closed sets are \( \mu \)-measurable, then \( E \) is \( \mu \)-measurable and
\[ \mu(E) = \sup_{C \in K(X); C \subseteq E} \mu(C) \]
since in this case \( E \) is Souslin \( \mathcal{F}(X) \). (See e.g. Saks [4].)

4.6. Remark. In the proof of Theorem 4.2 we do not use all the properties of a measure \( \mu \) in \( M(X, E) \). Part (iii) in the definition of \( M(X, E) \) may be replaced by
(iii)' if \( A \) is compact, \( A \subseteq E \), \( \epsilon > 0 \) then there exists an open set \( U \) such that \( A \subseteq U \) and \( \mu(U) \leq \mu(A) + \epsilon \) without affecting the validity of 4.2.

For the proof of 4.3 however (iii)' is not enough, but we may replace (iii) by
(iii)'' if \( A \) is compact, \( A \subseteq E \), \( \epsilon > 0 \), \( T \subseteq X \) and \( \mu(T) < \infty \) then there exists an open set \( U \) such that \( A \subseteq U \) and \( \mu(T \cap U) \leq \mu(T \cap A) + \epsilon \).
Part (iii) will be satisfied if we have (iii)' and, for every compact
$A \subseteq E$, $\mu(A) < \infty$, or if we have
(iii)'' if $A$ is compact, $A \subseteq E$, $\varepsilon > 0$ then there exists an open set $U$
such that $A \subseteq U$ and $\mu(U - A) < \varepsilon$.

Bibliography

   pp. 131–295.
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