PROOF OF A CONJECTURE OF ROUTLEDGE

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According to Routledge [1], every g.r. (general recursive) function of one variable can be expressed as $g(\phi(a))$. Here $\phi(a)$ is a p.r. (primitive recursive) function and $g(a)$ is a function defined by the schema

\[ g(n) = m, \]

\[ g(a) = h(a, g(\delta(a))), \quad \text{for } a \neq n \]

where $h(a, b)$ and $\delta(a)$ are p.r. functions and $\delta(a) < a$ for $a \neq n$ in a well-ordering, of order type $\omega$, of the natural numbers with $n$ as the first element. Routledge also conjectured that not every g.r. function can be expressed as $\phi(g(a))$ [1].

The purpose of this note is to give a proof for this conjecture by actually constructing a g.r. function $\psi(a)$ and then showing that for any p.r. function $\phi(a)$ and any function $g(a)$ defined by the preceding schema, $\psi(a)$ is not identically equal to $\phi(g(a))$. An argument for which $\psi(a) \neq \phi(g(a))$ can be actually found by using the method in the proof, provided that the well-ordering of type $\omega$ involved in the definition of $g(a)$ is itself constructively given. In this sense, the proof can be regarded as a constructive one.

**Definition.** A finite sequence of natural numbers

\[ a_0, a_1, \ldots, a_r \]

is called a w.o. (well-ordered) segment of rank $p$, if and only if $r > 0$, $a_r = p$, $a_i \neq a_j$ for $i \neq j$, and $a_i < a_r$ for $i < r$.

**Proof of the Conjecture.** Let $\varphi_1(y, a)$ and $\varphi_2(y, a, b)$ be two enumerating functions of p.r. functions of one variable and two variables respectively [2]. Let $\tau(a, b, c, d, e), \tau_1(a), \tau_2(a), \tau_3(a), \tau_4(a), \tau_5(a)$ be six p.r. functions such that for any five given numbers $a, b, c, d, e$, a unique number $x$ exists so that $\tau_1(x) = a, \tau_2(x) = b, \tau_3(x) = c, \tau_4(x) = d, \tau_5(x) = e$ and $\tau(a, b, c, d, e) = x$. (These functions can be constructed readily by using the p.r. functions $\sigma(a, b), \sigma_1(a), \sigma_2(a)$ as defined by Peter [3].) Now, let a g.r. function $\psi(a)$ be constructed as follows:

The first step. Put $\psi(0) = 0$.

The $(p + 1)$th step ($p > 0$). There are obviously only a finite number

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of w.o. segments of rank \( p \). Let this finite number be \( \pi(p) \). Consider any one of such segments \( a_0, a_1, \cdots, a_r \). We shall determine a number \( x \) from this segment and then use this number for the evaluation of \( \psi(p) \). Let \( x \) be the least number \( y \) not being used at the previous steps, such that \( y \) satisfies the conditions:

(i) \( \tau_1(y) = a_0 \),
(ii) \( \phi_1(\tau_6(y), a_i) = a_{i-1}, \) for \( i = 1, \cdots, r \).

It is noted that this number \( x \) can always be found. For, obviously there exists a number \( u^* \) such that the p.r. function \( \phi_1(u^*, a) \) has the property \( \phi_1(u^*, a_i) = a_{i-1} \) for \( i = 1, \cdots, r \); then the expression \( \tau(a_0, l, s, t, u^*) \) with \( l, s, t \) as parameters gives infinitely many numbers all satisfying the conditions (i) and (ii) when one of the parameters, say, \( l \) runs over all the natural numbers.

We use the number \( x \) to define a partial recursive function \( k(a) \) by

\[
k(a) = \phi_1(\tau_3(x), g(a)),
\]

\[
g(\tau_1(x)) = \tau_2(x),
\]

\[
g(a) = \phi_2(\tau_4(x), a, \phi(\tau_6(x), a))), \) for \( a \neq \tau_1(x).\)

Since \( p = a_r \) and \( x \) satisfies the conditions (i) and (ii), then it is seen that the value of \( k(a) \) for \( a = p \) is uniquely determined by the above equations and can be evaluated in a finite number of steps.

Since there are \( \pi(p) \) w.o. segments of rank \( p \), then we can determine \( \pi(p) \) numbers \( x_{p,1}, \cdots, x_{p,r} \) and then use them to define \( \pi(p) \) partial recursive functions \( k_{p,1}(a), \cdots, k_{p,r}(a) \) respectively in a similar way as described above. To conclude this step, we let

\[
\psi(p) = 1 + k_{p,1}(p) + \cdots + k_{p,r}(p).
\]

The function \( \psi(a) \) as constructed above is effectively calculable for each argument \( a \) and consequently \( \psi(a) \) is a g.r. function. (See, for example, [4].)

Let \( \phi(a) \) be any p.r. function and \( g(a) \) be any function defined at the beginning of this note. A number \( w \) is called a favorable argument of the function \( \phi(g(a)) \), if for any number \( i > 0, \delta^i(w) < w \), if \( \delta^i(w) < w \). We shall show in the following that there are infinitely many such favorable arguments and among them there is at least one argument for which \( \psi(a) \neq \phi(g(a)) \).

Given any natural number \( q \), we find the number \( k \) such that \( q \) is the \( k \)th element in the ordering \( < \). Find the greatest number \( z \) among the first \( k \) elements in the ordering \( < \) and again find the greatest number \( w \) among the \( (k+1) \)th, \( \cdots, \) \( (k+z+2) \)th elements in the
ordering $<$. If $<$, the well-ordering of type $\omega$, is constructively given, the numbers $k$, $z$ and also $w$ can be actually found. It can be easily verified that $q < w$, $n < w$ and that for any $y$, $y < w$, if $y < w$. Since $\delta(a) < a$ for $a \neq n$, then for any $i > 0$, $\delta^i(w) < w$, if $\delta^i(w) < w$. Consequently $w$ is a favorable argument of $\phi(g(a))$. Since $w$ is greater than the arbitrarily given number $q$, then the number of such favorable arguments is infinite.

We find three numbers $s$, $t$, $u$, such that $\varphi_1(s, a) = \phi(a)$, $\varphi_2(t, a, b)$ $= h(a, b)$ and $\varphi_1(u, a) = \delta(a)$. Let $x_0 = \tau(n, m, s, t, u)$. Suppose $w$ be any favorable argument of $\phi(g(a))$. Since $n$ is the first element in the ordering $<$, we can find a number $r$ which is the least number $i$ such that $\delta^i(w) = n$. Then the sequence $\delta^i(w), \ldots, \delta(w)$, $w$ is a w.o. segment of rank $w$ with its initial term $\delta^i(w) = n$. Denote this w.o. segment by $a_0, \ldots, a_r$. Then we have $\tau_1(x_0) = n = a_0$ and $\delta(a_i) = \varphi_1(\tau_i(x_0), a_i) = a_{i-1}$ for $i = 1, \ldots, r$. Hence $x_0$ satisfies the conditions (i) and (ii). According to the recipe for the construction of $\psi(a)$, the arbitrarily given favorable argument $w$ has the property that if $x_0$ is used neither at the $(w+1)$th step nor at any previous step, then there exists a number which is used at the $(w+1)$th step and is less than $x_0$. Among the infinitely many favorable arguments we can find $x_0 + 1$ of them, say, $w_1, \ldots, w_{x_0+1}$. It can not be that for every $j$ $(1 \leq j \leq x_0 + 1)$, $x_0$ is used neither at the $(w_j + 1)$th step nor at any other step preceding to it. For if it were the case, there would be $x_0 + 1$ distinct numbers all less than $x_0$. This is impossible. Hence there must be a step, say, the $(p+1)$th step (where $p \leq$ some $w_j$) at which $x_0$ is used for the evaluation of $\psi(p)$. This number $p$ is, of course, still a favorable argument of $\phi(g(a))$.

The number $x_0$ is, then, one among the $\pi(p)$ numbers $x_{p,1}, \ldots, x_{p,r}$ and is used to define one of the $\pi(p)$ partial recursive functions, say, $k_{p,i}(a)$. By definition, $\psi(p)$ is greater than $k_{p,i}(p)$ at least by 1. In fact, $k_{p,i}(a)$ is just the same function as $\phi(g(a))$. Thus $\psi(a)$ is not identically equal to $\phi(g(a))$. This completes the proof of the conjecture of Routledge.

References


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