Remark. If $X_1, X_2$ are two continuous curves and $\rho_i: X_i \to I$, $i = 1, 2$, are monotone mappings onto, then $S(X_1, X_2, \rho_1, \rho_2)$ need not be locally connected.

References


The University of Zagreb, Zagreb, Yugoslavia

A CLAN WITH ZERO WITHOUT THE FIXED POINT PROPERTY

HASKELL COHEN

There is a conjecture due to A. D. Wallace that a clan (i.e., a compact, connected, topological semigroup with identity element) with a zero element has the fixed point property. This is related to another conjecture of Wallace that a compact connected topological lattice has the fixed point property [4]. A proof of the latter conjecture for the finite dimensional case has recently been given by Dyer and Shields [1]. There is an example due to Kinoshita [2] of a contractable continuum without the fixed point property. The purpose of this note is to exhibit a multiplication which will make Kinoshita's example into a clan with zero, and, thus, provide a counter example to the first conjecture above.

We exhibit first a result which seems to be rather generally known, but which, to the author's knowledge, does not appear in print.

Lemma. Suppose $S$ is a topological semigroup, and $f$ is an open or closed map taking $S$ onto $T$, a Hausdorff space. Suppose further that $f(a) = f(b)$ and $f(c) = f(d)$ implies $f(ac) = f(bd)$. Then $T$ can be given a multiplication which makes it a topological semigroup and which makes $f$ a homomorphism.

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Proof. For \( t_1 \) and \( t_2 \) in \( T \) we define \( t_1 \cdot t_2 \) as follows. Let \( a_1 \) and \( a_2 \) be elements of \( S \) such that \( f(a_1) = t_1 \) and \( f(a_2) = t_2 \). Let \( t_1 \cdot t_2 = f(a_1 a_2) \). It is easily seen that the multiplication is well defined and associative, and that \( f \) is a homomorphism. The only item which remains to be checked is the continuity of the multiplication. Let \( m: T \times T \to T \) be defined by \( m(t_1, t_2) = t_1 \cdot t_2 \). We need only show that \( m \) is continuous. If \( f \) is an open map, let \( P \) be an open set in \( T \) (if \( f \) were closed, we would, of course, take \( P \) closed). Then \( m^{-1}(P) = \{ (t_1, t_2) \mid t_1 \cdot t_2 \in P \} = \{ (t_1, t_2) \mid f[f^{-1}(t_1) f^{-1}(t_2)] \in P \} \) is seen to be open since \( f \) and multiplication in \( S \) are continuous and \( f \) is open. Thus \( m \) is continuous completing the proof.

Now let

\[
A = \{(r, \theta, z) \mid 0 \leq r < 1, z = 0\},
\]

\[
B = \left\{(r, \theta, z) \mid r = \frac{2}{\pi} \tan^{-1} \theta, \theta \geq 0, 0 \leq z \leq 1\right\},
\]

and

\[
C = \{(r, \theta, z) \mid r = 1, 0 \leq z \leq 1\},
\]

where \( r, \theta, \) and \( z \) represent the usual cylindrical coordinates in three space. Let \( K = A \cup B \cup C \), then \( K \) is the continuum of the example of Kinoshita. For a continuous function on \( K \) to \( K \) which moves every point, the reader is referred to Kinoshita [2].

Let \( D \) be the projection of \( B \cup C \) into the \( z=0 \) plane. Define a multiplication \( \circ \) on \( D \) by

\[
(r_1, \theta_1) \circ (r_2, \theta_2) = \left[ \max \left\{ r_1, r_2, \frac{2}{\pi} \tan^{-1} (\theta_1 + \theta_2) \right\}, \theta_1 + \theta_2 \right].
\]

It is easy to see that \( \circ \) is associative and continuous and that \((0, 0)\) is an identity element. It, perhaps, should be mentioned that \( D \), in slightly different form, is a rather well known clan (see e.g. the example on page 286 of [3]).

Now let \( I \) be the interval \([0, 1]\) with the usual multiplication, and let \( E = D \times I \) with the coordinatewise multiplication; i.e., \( (r_1, \theta_1, z_1) \cdot (r_2, \theta_2, z_2) = [(r_1, 0_1) \circ (r_2, 0_2), z_1 z_2] \).

Let \( f: E \to R^3 \) be defined by

\[
f(r, \theta, z) = (r, \theta, z) \text{ if } r \leq 2z
\]

and

\[
f(r, \theta, z) = (2z, \theta, z) \text{ if } r \geq 2z.
\]
Let $F = f(E)$. It is easily seen that $f$ is continuous and, since $E$ is compact, that $f$ is closed. Let $p_i$ be the points $(r_i, \theta_i, z_i)$, $i = 1, 2, 3, 4$, and suppose $f(p_1) = f(p_2)$ and $f(p_3) = f(p_4)$. We want to show $f(p_1 \cdot p_2) = f(p_3 \cdot p_4)$. If $p_1 = p_2$ and $p_3 = p_4$, the result is clear. Hence suppose, say, $p_1 \neq p_2$. Then $f(p_1) = f(p_2)$ implies $r_1 \geq 2z_1$, $r_2 \geq 2z_2$, $\theta_1 = \theta_2$, and $z_1 = z_2$. Now the $r$ coordinate of $p_1 \cdot p_3 \geq r_1 \geq 2z_1 \geq 2z_1 z_3$ and similarly the $r$ coordinate of $p_2 \cdot p_4 \geq 2z_2 z_4$. Hence $f(p_1 p_3) = (2z_1 z_3, \theta_1 + \theta_3, z_1 z_3)$ $= (2z_2 z_4, \theta_2 + \theta_4, z_2 z_4) = f(p_2 p_4)$. Thus there is induced on $F$ the multiplication described in the lemma. With respect to this multiplication the points $(0, 0, 1)$ and $(0, 0, 0)$ are respectively the identity element and zero of $F$. Moreover it is clear that $F$ is homeomorphic to $K$. For example, the function $h$ defined by $h(r, \theta, z) = (r, \theta, (1 - r/2)z + r/2)$ takes $K$ homeomorphically onto $F$. Applying the lemma again, $h^{-1}$ induces a multiplication which makes $K$ a clan with zero as was to be shown.

REFERENCES