POWER SERIES WITH GAPS

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If every qth term in a power series $f(z) = a_0 + a_1 z + \cdots$ with finite radius of convergence is missing then $f(z)$ has at least two singularities on its circle of convergence (Mandelbrojt [1]). I proved more generally [2, Theorem 1):

If $a_m = 0$ for all $m \equiv \tau_1, \cdots, \tau_k \pmod{q}$, $k \leq 3$, where $\tau_1, \cdots, \tau_k$ for every divisor $q'$ of $q$ belong to $\max(k, q')$ different residue classes, then $f(z)$ has at least $k + 1$ singularities on its circle of convergence.

I wish to show here that this statement is not true for $k = 4$ and $k \geq 6$. The case $k = 5$ remains open. It would be of interest to determine the minimum number $\nu$ of singularities for given $k$, as well as for given $k$ and $q$. For prime $q$ it is known [3] that $\nu \geq k + 1$.

The pertinent counterexample is

$$f(z) = \frac{1 - z^\alpha}{1 - z^{\alpha \beta}} = 1 - z^\alpha + z^{\alpha \beta} - z^{\alpha + \alpha \beta} + \cdots,$$

with $q = \alpha \beta$ and $\alpha \beta - \alpha$ singularities on the unit circle. Let $\tau_1, \cdots, \tau_k$ be the $q - \gamma$ numbers $0, \cdots, q - 1$ after deletion of $0, \cdots, \gamma - 2$ and $\alpha$, where $2 \leq \gamma \leq \alpha$. If $\beta \geq 3$ then, for every proper divisor $q'$ of $q$, all residue classes are represented by $q - q', \cdots, q - 1$. However, $\alpha \beta - \alpha \leq k$.

For given $k$, the choice of $\alpha$ (not exceeding $k/2$ and not dividing $k + 1$) determines $\beta$ and $\gamma$ uniquely. Set $\alpha = \lceil k/2 \rceil$, $\beta = 3$, $\gamma = k/2$ or $(k - 3)/2$; then $\gamma \geq 2$ excludes only $k = 1, 2, 3, 5$.

REFERENCES


2. T. S. Motzkin, Bemerkung über Singularitäten gewisser mit Lücken behafteter Potenzreihen, Math. Ann. vol. 109 (1933) pp. 95–100; in Theorem 1, Condition 1 is to be taken in its stronger meaning: “in” classes does not imply exhaustion of those classes.


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