POWER SERIES WITH GAPS

T. S. MOTZKIN

If every qth term in a power series \( f(z) = a_0 + a_1z + \cdots \) with finite radius of convergence is missing then \( f(z) \) has at least two singularities on its circle of convergence (Mandelbrojt [1]). I proved more generally [2, Theorem 1]:

\[
\text{If } a_m = 0 \text{ for all } m \equiv \tau_1, \cdots, \tau_k \pmod{q}, \; k \leq 3, \text{ where } \tau_1, \cdots, \tau_k \text{ for every divisor } q' \text{ of } q \text{ belong to } \max(k, q') \text{ different residue classes, then } f(z) \text{ has at least } k + 1 \text{ singularities on its circle of convergence.}
\]

I wish to show here that this statement is not true for \( k = 4 \) and \( k \geq 6 \). The case \( k = 5 \) remains open. It would be of interest to determine the minimum number \( \nu \) of singularities for given \( k \), as well as for given \( k \) and \( q \). For prime \( q \) it is known [3] that \( \nu \geq k + 1 \).

The pertinent counterexample is

\[
f(z) = \frac{1 - z^{a}}{1 - z^{\alpha}} = 1 - z^{\alpha} + z^{\alpha \beta} - z^{\alpha + \alpha \beta} + \cdots,
\]

with \( q = \alpha \beta \) and \( \alpha \beta - \alpha \) singularities on the unit circle. Let \( \tau_1, \cdots, \tau_k \) be the \( q - \gamma \) numbers \( 0, \cdots, q-1 \) after deletion of \( 0, \cdots, \gamma - 2 \) and \( \alpha \), where \( 2 \leq \gamma \leq \alpha \). If \( \beta \geq 3 \) then, for every proper divisor \( q' \) of \( q \), all residue classes are represented by \( q - q', \cdots, q - 1 \). However, \( \alpha \beta - \alpha \leq k \).

For given \( k \), the choice of \( \alpha \) (not exceeding \( k/2 \) and not dividing \( k + 1 \)) determines \( \beta \) and \( \gamma \) uniquely. Set \( \alpha = \lfloor k/2 \rfloor \), \( \beta = 3 \), \( \gamma = k/2 \) or \( (k-3)/2 \); then \( \gamma \geq 2 \) excludes only \( k = 1, 2, 3, 5 \).

References


2. T. S. Motzkin, Bemerkung über Singularitäten gewisser mit Lücken behafteter Potenzreihen, Math. Ann. vol. 109 (1933) pp. 95–100; in Theorem 1, Condition 1 is to be taken in its stronger meaning: “in” classes does not imply exhaustion of those classes.


University of California, Los Angeles

Presented to the Society, April 23, 1960; received by the editors August 10, 1959.

\( ^1 \) Sponsored, in part, by the Office of Naval Research. Reproduction in whole or in part is permitted for any purpose of the United States Government.