ALMOST UNIFORM CONVERGENCE VERSUS
POINTWISE CONVERGENCE

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In many an example of a function space whose topology is the
topology of almost uniform convergence it is observed that the same
topology is obtained in a natural way by considering pointwise con-
vergence of extensions of the functions on a larger domain [1; 2].
This paper displays necessary conditions and sufficient conditions
for the above situation to occur.

Consider a linear space $G(S, F)$ of functions with domain $S$ and
range in a real or complex locally convex linear topological space $F$.
Assume that there are sufficient functions in $G(S, F)$ to distinguish
between points of $S$. Let $S_\beta$ denote the closure of the image of $S$
in the cartesian product space $\times \{g(S) : g \in G(S, F)\}$. Theorems 4.1
and 4.2 of reference [2] give the following theorem.

**Theorem.** If $g(S)$ is relatively compact for every $g$ in $G(S, F)$, then
pointwise convergence of the extended functions on $S_\beta$ is equivalent to
almost uniform convergence on $S$.

When almost uniform convergence is known to be equivalent to
pointwise convergence on a larger domain the situation can usually
be converted to one of equivalence of the two modes of convergence
on the same domain by means of Theorem 4.1 of [2]. In the new
formulation the following theorem is applicable.

In preparation for the theorem, let $B(S, R)$ denote all bounded
real valued functions on $S$ which are uniformly continuous for the
uniformity which $G(S, F)$ generates on $S$. $G(S, F)$ will be called a
**full linear space** if for every $f$ in $B(S, R)$ and every $g$ in $G(S, F)$ the
function $fg$ obtained from their pointwise product is a member of
$G(S, F)$. $(S, G)$ denotes $S$ with the weakest topology such that each
$g$ in $G(S, F)$ is continuous, while $(S, G) \cup \{\infty\}$ denotes the one point
compactification of $(S, G)$.

**Theorem.** If $G(S, F)$ is a full linear space in which pointwise con-
vergence and almost uniform convergence are equivalent then $(S, G)$ is

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either (i) compact, or (ii) locally compact and each \( g \) in \( G(S, F) \) can be continuously extended over \( (S, G) \cup \{ \infty \} \) such that \( g(\infty) = 0 \).

**Proof.** Let \( CS \) be the compactification of \( (S, G) \) obtained by taking the closure of the image of \( S \) in \( \times \{ f(S) : f \in B(S, R) \} \). Assume \( (S, G) \) is not compact and that there is an \( s_0 \) in \( CS - S \), a net \( \{ s_\alpha \} \) in \( S \), and a function \( g \) in \( G(S, F) \) such that the net \( \{ s_\alpha \} \) converges to \( s_0 \) but the net \( \{ g(s_\alpha) \} \) does not converge to 0. There exists a net \( \{ f_\alpha \} \) in \( B(S, R) \) converging pointwise on \( S \) to the function which is identically one while all continuous extensions of the members of the net have the value zero at \( s_0 \). This gives a contradiction in that the net \( \{ f_\alpha g \} \) in \( G(S, F) \) converges pointwise to \( g \) but not almost uniformly. Thus when \( (S, G) \) is not compact each \( g \) in \( G(S, F) \) can be continuously extended over \( CS \) so that \( g(s) = 0 \) for every \( s \) in \( CS - S \). Since \( CS \) is the Hausdorff completion of \( (S, G) \), there is only one point in \( CS - S \). Thus \( (S, G) \) is locally compact and the proof completed.

**Corollary.** A uniform space \( E \) is compact if and only if pointwise convergence and almost uniform convergence are equivalent in \( C(E, R) \), all uniformly continuous real valued functions on \( E \).

**Proof.** If \( E \) is compact, see Theorem 4.2 of [2]. For the converse, observe that each \( f \) in \( C(E, R) \) is bounded, Theorem 1.3 of [2]. Thus \( fg \) is in \( C(E, R) \) whenever \( f \) and \( g \) are, and \( C(E, R) \) is a full linear space. Since \( C(E, R) \) contains nonzero constant functions, conclusion (ii) of the second theorem is not possible and \( E \) must be compact.

**Bibliography**


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