NOTE ON THE HOMOTOPY PROPERTIES OF THE COMPONENTS OF THE MAPPING SPACE \( X^{S^p} \)

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1. Introduction. Let \( X \) be a topological space and \( S^p \) be the polarized \( p \)-sphere with a fixed pole \( y_0 \). Following G. W. Whitehead \([10]\), we shall denote by \( G^p(X) \) the mapping space \( X^{S^p} \), which is the totality of (continuous) maps of \( S^p \) into \( X \) endowed with compact-open topology. Let \( \pi: G^p(X) \rightarrow X \) be defined by \( \pi(f) = f(y_0) \) (\( f \in G^p(X) \)), and let \( \pi_*(X, x) = \pi^{-1}(x) \) for each \( x \in X \). Consider now the mapping space \( B(X) \) consisting of all the maps of \( y_0 \) into \( X \). There is a natural map \( \varphi: G^p(X) \rightarrow B(X) \) defined by \( \varphi(f) = f|y_0 \) for every \( f \in G^p(X) \). It is well known (cf. \([3, \text{pp. 83-84}]\)) that \( \varphi \) has the path lifting property. Clearly, the space \( X \) can be identified with \( B(X) \) in a natural way. The map \( \pi \) is then identified with \( \varphi \). Consequently \( \pi: G^p(X) \rightarrow X \) is a fibre map of \( G^p(X) \) onto \( X \) having the absolute covering homotopy property \([3, \text{p. 82}]\). For each \( x \in X \), the fibre in \( G^p(X) \) over \( x \) is \( F^p(X, x) \). The arc components of \( F^p(X, x) \) are elements of the \( p \)th homotopy group \( \pi_p(X, x) \) of \( X \) at \( x \). Denote by \( G^p_\alpha(X) \) the arc component of \( G^p(X) \) which contains \( \alpha = F^p_\alpha(X, x) \in \pi_*(X) \) (cf. \([10]\)). If \( X \) is arcwise connected, then \( G^p_\alpha(X) \) is also a fibre space over \( X \). The restriction \( \pi_\alpha = \pi|G^p_\alpha(X) \) is a fibre map of \( G^p_\alpha(X) \) onto \( X \). The homotopy properties of the various components \( G^p_\alpha(X) \) of \( G^p(X) \) have been studied by M. Abe (Jap. J. Math. vol. 16 (1940) pp. 169-176), G. W. Whitehead \([10]\) and S. T. Hu \([2]\). The present note may be regarded as a continuation of these studies.

2. \( H \)-space and \( H_\ast \)-space. In what follows, we shall denote \( G^p(X) \) by \( G^p \) and \( F^p(X, x) \) by \( F^p \) whenever no confusion is likely to arise.

Let \( X \) be a topological space which admits a continuous multiplication \( \mu(x, x') = x \cdot x' \). If \( f: S \rightarrow X \) is a map of a space \( S \) into \( X \), we denote by \( x \cdot f \) the transformation defined by \( (x \cdot f)(s) = x \cdot f(s) \) for each \( s \in S \). Clearly \( x \cdot f \) is a map (i.e. it is continuous).

By an \( H \)-space we mean a topological space \( X \) with a given continuous multiplication which has a homotopy unit \( e \in X \) (see e.g. \([3, \text{pp. 80-81}]\)).
(2.1) Theorem. If $X$ is an arcwise connected $H$-space, then $G^p_\alpha(X)$ and $G^p_\beta(X)$ have the same homotopy type for arbitrary $\alpha$ and $\beta$ in $\pi_p(X)$, $\rho \geq 1$.

Proof. It suffices to prove that $G^p_\alpha(X)$ and $G^p_\beta(X)$ have the same homotopy type, for any $\alpha \in \pi_p(X)$. According to [10], it remains to prove that $G^p_\alpha(X)$ admits a (global) cross-section. Choose an element $f \in G^p_\alpha \cap F^p(X, e)$. Then $\pi_\alpha(f) = e$. Define $\phi: X \to G^p$ by $\phi(x) = x \cdot f$. Then $\phi(e) = e \cdot f \cong f \in G^p_\alpha$. Since $X$ is arcwise connected, we have $\phi: X \to G^p_\alpha$. Now, $\pi_\alpha(\phi(x)) = \pi_\alpha(x \cdot f) = x \cdot e$, therefore $\pi_\alpha \phi \cong \text{id}_X$. Since $\pi_\alpha$ has the absolute covering homotopy property, there exists a covering homotopy, in particular, there is a map $\psi: X \to G^p_\alpha$ such that $\pi_\alpha \psi = \text{id}_X$. This proves (2.1).

Following H. Wada [9], we call a topological space $X$ an $H_*$-space if the following conditions are satisfied:

(i) A continuous multiplication $\mu(x, x') = x \cdot x'$ is defined for each pair of elements $x, x'$ in $X$.

(ii) There is a fixed element $e$ in $X$, satisfying

$$x \cdot e = x,$$

for all $x \in X$.

(iii) To each $x \in X$, there is an inverse $x^{-1} \in X$, defined continuously by $x$, such that

$$x \cdot x^{-1} = e,$$

for all $x \in X$.

(iv) For each pair of elements $x, x'$ in $X$, we have

$$x^{-1} \cdot (x \cdot x') = x'.$$

With these conditions Wada was able to prove that

(ii') $e$ is unique,

(iii') $x^{-1}$ is uniquely defined by $x$ and $x^{-1} \cdot x = e$,

(v) $(x^{-1})^{-1} = x$ and, consequently, $x \cdot (x^{-1} \cdot x') = x'$, for arbitrary $x$ and $x'$ in $X$.

We remark that an $H_*$-space need not to be an $H$-space.

The following theorem resembles a construction of Wada [9], where he deals with mapping space of an $H_*$-space into itself.

(2.2) Theorem. Let $X$ be an $H_*$-space. The mapping space $G^p(X)$ is homeomorphic to $X \times F^p(X, e)$ for each $\rho \geq 1$.

Proof. Let $g \in G^p(X)$ be an arbitrary map of $S^p$ into $X$. Then $x \cdot g$ is defined and continuous. Hence $x \cdot g \in G^p(X)$. Clearly $g = e \cdot g = x^{-1} \cdot (x \cdot g) = x \cdot (x^{-1} \cdot g)$ for any $x \in X$. Let
\(\phi: G^p(X) \to X \times F^p(X, e),\)

and

\(\psi: X \times F^p(X, e) \to G^p(X),\)

be defined as follows: Let \(y_0\) be the pole of \(S^p\). For each \(g \in G^p(X)\), let \(g = g(y_0) \in X\). Then define

\[\phi(g) = (\hat{g}, \hat{g}^{-1} \cdot g), \quad (g \in G^p(X))\]

and

\[\psi(x, f) = x \cdot f, \quad (x \in X, f \in F^p(X, e)).\]

(A) \(\phi\) and \(\psi\) are bijective:

For any \(g \in G^p(X)\), we have

\[\psi \phi(g) = \psi(\hat{g}, \hat{g}^{-1} \cdot g) = g.\]

On the other hand,

\[\phi \psi(x, f) = \phi(x \cdot f) = ((x \cdot f)^-, ((x \cdot f)^-)^{-1} \cdot (x \cdot f))\]

\[= (x \cdot f, (x \cdot f)^{-1} \cdot (x \cdot f))\]

\[= (x, x^{-1} \cdot (x, f))\]

\[= (x, f).\]

Hence both \(\phi\) and \(\psi\) are one-to-one, onto.

(B) \(\phi\) and \(\psi\) are continuous:

Suppose \(K\) be a compact set in \(S^p\) and \(U\) an open set in \(X\). We shall denote by \((K, U)\) be the subset of \(G^p(X)\) consisting of all mappings which send \(K\) into \(U\). Let \(H\) be an arbitrary neighborhood of \((\hat{g}, \hat{g}^{-1} \cdot g)\). Then \(H \supset U_0 \times [(K_1, U_1) \cap \cdots \cap (K_n, U_n)]\) for some open sets \(U_0, U_1, \ldots, U_n\) in \(X\) and compact sets \(K_1, \ldots, K_n\) in \(S^p\). Denote \(g(K_i)\) by \(K_i'\), then \(K_i'\) is compact, \(i = 1, 2, \ldots, n\). Corresponding to each \(k_i' \in K_i'\), there exist open sets \(W_i'\) containing \(\hat{g}^{-1}\) and \(V_i'\) containing \(k_i'\) such that \(W_i' \cdot V_i' \subseteq U_i\), since the multiplication in \(X\) is continuous. The collection \(\{V_i'\}\) forms an open covering of \(K_i'\). There is a finite subcovering \(\{V_i', \ldots, V_i'^{m_i}\}\) of \(K_i'\). Let \(W_i = \cap_{m=1}^{m_i} W_i^{m_i}\) and \(V_i = \cup_{m=1}^{m_i} V_i^{m_i}\). Then \(W_i\) is an open neighborhood of \(\hat{g}^{-1}\); \(V_i\) is an open neighborhood of \(K_i\) and \(W_i, V_i \subseteq U_i\).

Let \(N = (y_0, U_0 \cap W_1^{-1} \cap \cdots \cap W_n^{-1}) \cap (K_1, V_1) \cap \cdots \cap (K_n, V_n)\), where \(W_i^{-1}\) denotes, of course, the set \(\{w^{-1} | w \in W_i\}\). By the continuity of the inverse, \(N\) is a neighborhood of \(g\) in \(G^p\). It is now readily seen that \(\phi(N) \subseteq H\). This proves the continuity of \(\phi\).

Next, let \(U = (K_1, U_1) \cap (K_2, U_2) \cap \cdots \cap (K_n, U_n)\) be a basic open
neighborhood of $\psi(x, f) = x \cdot f$. Then $x \cdot f(K_i) \subset U_i$. By a similar argument as above, one proves that there exist open neighborhoods $W_i$ of $x$ and $V_i$ of $f(K_i)$ such that $W_i \cdot V_i \subset U_i$. Then

$$\psi[(W_1 \cap \cdots \cap W_n) \times ((K_1, V_1) \cap \cdots \cap (K_n, V_n) \cap F^p)] \subset U.$$  

Hence $\psi$ is continuous and the proof of (2.2) is completed.

(2.3) Corollary. If $X$ is an arcwise connected $H_*$-space, then $G_\alpha^p$ and $X \times F_\alpha^p$ are homeomorphic.

Proof. Since $X$ is arcwise connected, $G_\alpha^p$ is a fibre space over $X$. By replacing $G^p$ and $G_\alpha^p$ and $\pi$ by $\pi_\alpha$ in the proof of (2.2), we obtain that $G_\alpha^p$ is homeomorphic to $X \times \pi_\alpha^{-1}(e)$. Being a component, $G_\alpha^p$ is connected hence $\pi_\alpha^{-1}(e)$ contains only one component $F_\alpha^p$. This proves (2.3).

As a by-product of the proof of (2.3) we have:

(2.4) Corollary. Every arcwise connected $H_*$-space is $n$-simple, for $n \geq 1$.

(2.5) Corollary. If $X$ is an arcwise connected $H_*$-space, then $G_\alpha^p$ and $G_\beta^p$ have the same homotopy type for arbitrary $\alpha$ and $\beta$ in $\pi_p(X)$. Furthermore

$$\pi_q(G_\alpha^p) \cong \pi_{p+q}(X) + \pi_q(X), \quad (q \geq 1).$$

Proof. Since G. W. Whitehead [10] proved that $F_\alpha^p$ and $F_\beta^p$ have the same homotopy type for any $\alpha$ and $\beta$ in $\pi_p(X)$, the first part of (2.5) follows from (2.3). The Hurewicz isomorphism $\pi_q(F_\alpha^p) \cong \pi_{p+q}(X)$ (cf. [10]) completes the proof.

(2.6) Corollary. Let $X = S^r$. Then $G_\alpha^p$ is homeomorphic to $S^r \times F_\alpha^p$ when $r = 1, 3$ or 7. Conversely, if $G_\alpha^p$ and $S^r \times F_\alpha^p$ have the same homotopy type when $r = 1, 3$ or 7, where $i_r \in \pi_r(S^r)$ is represented by the identity map $S^r \to S^r$.

Proof. This follows from Wada [8] and a recent result of Adams [1].

(2.6) Proposition. If $X$ is a $H$-space, then for each $\alpha \in \pi_p(X)$,

$$\pi_q(G_\alpha^p)/\pi_{p+q}(X) \cong \pi_q(X),$$

where $\pi_{p+q}(X)$ is, of course, imbedded in $\pi_q(G_\alpha^p)$ isomorphically.

Proof. According to G. W. Whitehead [10] (see also [11]), we have the following diagram:
\[ \cdots \to \pi_q(G^n_a) \overset{i_*}{\to} \pi_q(G^p_a) \overset{j_*}{\to} \pi_q(G^p_{a!} F^n_a) \overset{\partial}{\to} \pi_{q-1}(F^n_a) \to \cdots \]

\[
\begin{array}{ccc}
\pi_{p+q}(X) & \rightarrow & \pi_{p+q-1}(X), \\
\downarrow & & \downarrow \\
\pi_q(X) & \rightarrow & \pi_{p+q-1}(X),
\end{array}
\]

where \(\pi_*\) denotes the isomorphism induced by the projection \(\pi\), \(H\) denotes the Hurewicz isomorphism and \(\rho_a\) is defined by \(\rho_a(\beta) = -[\alpha, \beta]\). Since \(\rho_a\) is always trivial when \(X\) is an \(H\)-space, (2.7) follows from the exactness of the sequence.

3. The sphere \(S^r\). Let \(X = S^r\), an \(r\)-sphere, then we have the following exact sequence

\[(3.1) \quad \cdots \to \pi_{p+q}(S^r) \overset{\mu}{\to} \pi_q(G^n_a) \overset{\nu}{\to} \pi_q(S^r) \overset{\rho_a}{\to} \pi_{p+q-1}(S^r) \overset{\mu}{\to} \cdots \]

The following propositions are fairly obvious.

(3.2) **Proposition.** Let \(X = S^r\) and \(a \in \pi_p(S^r)\). Since \(\pi_q(S^r) = 0\) for \(q < r\) we have

\[\pi_q(G^n_a) \cong \pi_{p+q}(S^r), \quad (q < r - 1).\]

(3.3) **Corollary.** \(\pi_1(G^n_a) \cong \mathbb{Z}_2\) for \(r \geq 3\).

Since \(\pi_{r+2}(S^r) \cong \mathbb{Z}_2\), for \(r \geq 3\), we have

(3.4) **Corollary.** \(\pi_1(G^{r+1}) \cong \mathbb{Z}_2\), \(r \geq 3\).

Denote the image of \(\rho_a : \pi_q(S^r) \to \pi_{p+q-1}(S^r)\) by \(J^{p+q-1}_a\) and the kernel of \(\rho_a\) by \(K^n_a\). Denote the image of \(\mu : \pi_{p+q}(S^r) \to \pi_q(G^n_a)\) by \(P_a\). Then

(3.5) **Proposition (Hu) [2].** For \(X = S^r\) and \(a \in \pi_p(S^r)\)

(a) \(\pi_q(G^p) / P^q_a \cong K^q_a\), \(q > 1\),

(b) \(\pi_{p+q}(G^p) / J^{p+q}_a \cong P^q\), \(q > 1\),

(c) \(\pi_{r-1}(G^n_a) \cong \pi_{p+r-1}(S^r) / J^{p+r-1}_a\),

(d) \(\pi_{r+3}(G^p)\) has a subgroup \(F^{r+3}_a \cong \pi_{p+r+3}(S^r)\), \(r \geq 6\),

(e) \(\pi_{r+4}(G^p) \cong \pi_{p+r+4}(S^r) / J^{p+r+4}_a\), \(r \geq 6\).

Since for \(r \geq 7\), \(\pi_{r+4}(S^r) = \pi_{r+5}(S^r) = 0\). It follows that

(3.6) **Proposition.** If \(r > 7\), for each \(a \in \pi_p(S^r)\),

\[\pi_{r-1}(G^p) \cong \pi_{r-2}(G^p) \cong \cdots \approx 0.\]

And,
(3.7) Proposition. For $r \geq 7$, $\alpha \in \pi_p(S^r)$,

$$\pi_{r+6-p}(G^p) \approx \pi_{r+6-p}(S^r).$$

We now proceed to prove the main theorem of this section. Consider the following sequence

$$\pi_r(S^r) \xrightarrow{\rho_\alpha} \pi_{2r-1}(S^r) \xrightarrow{E} \pi_{2r}(S^{r+1}),$$

where $E$ denotes the Freudenthal suspension. By the delicate suspension theorem, the kernel of $E$ is a cyclic subgroup generated by $[\iota_r, \iota_r]$. If $r$ is even, it is infinite cyclic; if $r$ is odd $\neq 1, 3, 7$, it is cyclic of order 2.

(3.9) Lemma (Hu). For $X = S^2$ and $\alpha \in \pi_2(S^2)$, we have

$$\pi_1(G^2_\alpha) \approx Z_{2m},$$

where $m$ is the absolute value of the degree of $\alpha$.

Proof. Since $\pi_{2r}(S^{r+1}) = \pi_4(S^0) \approx Z_2$. From (3.8) $\pi_3(S^2)/\text{Ker } E \approx Z_2$. Let $\gamma$ be a generator of the free cyclic group $\pi_3(S^2)$. Then $[\iota_2, \iota_2] = \pm 2$. We can choose $\gamma$ so that $[\iota_2, \iota_2] = -2\gamma$. Let $\alpha \in \pi_2(S^2)$. By linearity of the Whitehead product $\rho_\alpha(\iota_2) = -[\alpha, \iota_2] = -m[\iota_2, \iota_2] = 2m\gamma$. In other words $J_\alpha^2$ is generated by $2m\gamma$. From (3.5(c)), we have $\pi_1(G^2_\alpha) \approx Z_{2m}$. This proves (3.9).

(3.10) Lemma. For $X = S^4$ and $\alpha \in \pi_4(S^4)$, we have

$$\pi_3(G^4_\alpha) \approx Z_{24m} + Z_{12},$$

where $m$ is the absolute value of the degree of $\alpha$.

Proof. $\pi_{2r}(S^{r+1}) = \pi_8(S^0) \approx Z_{24}$ and $\pi_{2r-1}(S^r) = \pi_7(S^4) \approx Z + Z_{12}$. One generator of $\text{Ker } E$ is determined as follows:

From a theorem of characteristic map [5, p. 121], that

$$[\iota_4, \iota_4] = 2[q] - \varepsilon E[\xi],$$

where $\varepsilon = \pm 1$ depends on the convention of orientation, $[q]$ denotes the homotopy class of the Hopf map $q: S^7 \to S^4$ and $[\xi]$ a generator of $\pi_6(S^6)$ represented by the characteristic map $\xi: S^6 \to S^4$ of the fibre bundle $Sp(2)$ over $S^7$ with $Sp(1)$ as fibre. Hence in $\pi_7(S^4)$ we have

$$E^2[\xi] = \varepsilon E[q],$$

($E^2$ denotes the iterated suspension). This implies that $\pi_8(S^6)$ has $E[q]$ as a generator. Hence

$$\pi_7(S^4)/\text{Ker } E \approx Z_{24} + Z_{12}.$$
A similar argument as used in (3.9) yields

\[ \pi_3(G \alpha) \cong Z_{24m} + Z_{12}. \]

(3.11) **Lemma.** For \( X = S^6 \) and \( \alpha \in \pi_6(S^6) \), we have

\[ \pi_6(G \alpha) \cong Z_m, \]

where \( m \) is the absolute value of the degree of \( \alpha \).

**Proof.** Since \( \pi_{2r}(S^{r+1}) = \pi_{12}(S^7) = 0 \) and \( \pi_{2r-1}(S^6) = \pi_{11}(S^6) \cong Z \). Ker \( E = J_\alpha \). Hence we can choose the generator \( \gamma \) of \( \pi_{11}(S^6) \) such that \( \gamma = -[\iota_6, \iota_6] \), consequently \( \rho_\alpha(\iota_6) = m \gamma \), or \( \pi_6(G \alpha) \cong Z_m \) by (3.4(c)).

(3.12) **Lemma.** For \( X = S^8 \) and \( \alpha \in \pi_8(S^8) \), we have

\[ \pi_7(G \alpha) \cong Z_{240m} + Z_{120}, \]

where \( m \) is the absolute value of the degree of \( \alpha \).

**Proof.** Since \( \pi_{2r}(S^{r+1}) = \pi_{16}(S^9) \cong Z_{240} \) and \( \pi_{2r-1}(S^8) = \pi_{15}(S^8) \cong Z + Z_{120} \). Since \( [\iota_6, \iota_6] = 2[q'] - eE[e'] \), where \([q']\) denote the homotopy class represented by the Hopf map \( q': S^{16} \rightarrow S^8 \) and \( \xi(\in \pi_{14}(S^7)) \) has nonzero Hopf invariant, we have

\[ E^2[e'] = 2eE[q']. \]

Using the same argument as in (3.10), one proves (3.12).

(3.13) **Lemma.** For \( X = S^{10} \) and \( \alpha \in \pi_{10}(S^{10}) \), we have

\[ \pi_9(G^{10}) \cong Z_m + Z_2 + Z_2 + Z_2, \]

where \( m \) denotes the absolute value of the degree of \( \alpha \).

(3.14) **Lemma.** For \( X = S^{12} \) and \( \alpha \in \pi_{12}(S^{12}) \), we have

\[ \pi_{11}(G^{12}) \cong Z_m + Z_8 + Z_27 + Z_7, \]

where \( m \) denotes the absolute value of the degree of \( \alpha \).

The proof of (3.13) follows from the table in Toda [6] the first row and a similar argument as before; for a proof of (3.14), one uses the third row of the above mentioned table.

(3.15) **Lemma.** For \( X = S^{14} \) and \( \alpha \in \pi_{14}(S^{14}) \), we have

\[ Z_{18}(G^{14}) \cong Z_m + Z_3, \]

where \( m \) denotes the absolute value of the degree of \( \alpha \).

**Proof.** Since \( \pi_{17}(S^{14}) \cong Z + Z_2 \) and \( \pi_{18}(S^{16}) \cong Z_3 \) and the suspension \( E \) sends \( Z \) into 0 in \( \pi_{18}(S^{14}) \) (Toda [6]). The proof is immediate.
(3.16) **Lemma.** For $X = S^r$, $\alpha \in \pi_r(S^r)$ and $r$ odd, $\not= 1, 3, 7$. Then

(a) $\pi_{r-1}(G^r) \approx \pi_{2r-1}(S^r)$ when $\alpha$ is of even degree,
(b) $\pi_{r-1}(G^r) \approx \pi_{2r-1}(S^r)/\mathbb{Z}_2$ when $\alpha$ is of odd degree.

**Proof.** It suffices to prove that there is a nonzero element in $J_{r-1}$ when $\alpha$ is of odd degree and $r \not= 1, 3, 7$. In fact, in this case $\rho_\alpha(\nu) \not= 0$. (3.16) follows.

(3.17) **Lemma (Hu).** Let $X$ be any space. If $\alpha, \beta \in \pi_p(X)$, $\alpha + \beta = 0$. Then $G^p_\alpha$ and $G^p_\beta$ are homeomorphic.

**Proof.** Let $\theta : S^p \to S^p$ be a homeomorphism which reverses the orientation and leaves the pole $y_0$ fixed. Then a homeomorphism $h$ of $G^p_\alpha$ onto $G^p_\beta$ is given by $h(f) = f \cdot \theta$ for each $f \in G^p_\alpha$.

(3.18) **Theorem.** Let $X = S^r$. Let $\alpha, \beta \in \pi_r(S^r)$. Then for $r = 2, 4, 6, 8, 10, 12, 14$, the components $G^r_\alpha$ and $G^r_\beta$ have the same homotopy type if and only if $\alpha = \pm \beta$. When $r$ is odd $\not= 1, 3, 7$, the components $G^r_\alpha$ and $G^r_\beta$ are of different homotopy type if $\deg \alpha = \deg \beta$ is odd.

**Proof.** The first part of the theorem follows from Lemmas (3.9) through (3.17). The remaining part follows from the fact that if $r$ is odd then $\pi_p(S^r)$ is finite for $p > n$ [4].

(3.19) **Corollary.** Let $X = S^r$ and $\alpha, \beta \in \pi_r(S^r)$ are of odd and even degree respectively. Then:

- $\pi_4(G^5_\alpha) = 0$, $\pi_4(G^5_\beta) \approx \mathbb{Z}_2$,
- $\pi_8(G^9_\alpha) \approx \mathbb{Z}_2 + \mathbb{Z}_2$,
- $\pi_8(G^9_\beta) \approx \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$,
- $\pi_{10}(G^{11}_\alpha) \approx \mathbb{Z}_2 + \mathbb{Z}_9$,
- $\pi_{10}(G^{11}_\beta) \approx \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_9$,
- $\pi_{12}(G^{13}_\alpha) = 0$,
- $\pi_{12}(G^{13}_\beta) \approx \mathbb{Z}_2$,
- $\pi_{14}(G^{15}_\alpha) \approx \mathbb{Z}_2 + \mathbb{Z}_2$ or $\mathbb{Z}_4$,
- $\pi_{14}(G^{15}_\beta) \approx \mathbb{Z}_4 + \mathbb{Z}_2$.

(3.20) **Proposition (Hu).** When $r$ is even and $\alpha \in \pi_r(S^r)$, $r \not= 0$. Then

$\pi_r(G^r_\alpha) \approx \pi_{2r}(S^r)/J_{2r}^1$.

**Proof.** Since $K^r_\alpha = 0$, the result follows from (3.4(a)) and (3.4(b)).

(3.21) **Proposition.** If $E : \pi_p(S^r) \to \pi_{p+1}(S^{r+1})$ is an injection, then for $q+s = p$ and $q > 1$

$\pi_q(G^s_\alpha)/\pi_p(S^r) \approx \pi_q(S^r)$,

where $\pi_p(S^r)$ is imbedded in $\pi_q(G^s_\alpha)$. 

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Proof. Since \( E[\alpha, \beta] = 0 \), \( J_{\alpha} \subset \text{Ker } E = 0 \). From (3.4)(a) and (b), 
\( \pi_q(G_{\alpha}^s) / \pi_{q+s}(S^r) \approx K_{\alpha}^r \). But \( K_{\alpha}^r = \pi_q(S^r) \). This proves (3.21).

For \( q < r \), \( \pi_q(S^r) = 0 \), we have \( \pi_q(G_{\alpha}^s) \approx \pi_p(S^r) \). This reduces to (3.2).

(3.22) Corollary. If \( q + s = p < 2r - 1 \), then 
\[ \pi_q(G_{\alpha}^s) / \pi_p(S^r) \approx \pi_q(S^r) \].

Proof. This follows from (3.21).

Bibliography


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