ABSOLUTE-VALUED ALGEBRAS

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An algebra $A$ over the real field $R$ is a vector space over $R$ which is closed with respect to a product $xy$ which is linear in both $x$ and $y$, and which satisfies the condition $\lambda(xy) = (\lambda x)y = x(\lambda y)$ for any $\lambda$ in $R$ and $x, y$ in $A$. The product is not necessarily associative. An element $e$ of the algebra $A$ is called a unit element if $ex = xe = x$ for any $x$ in $A$. Given any subset $B$ of $A$, $\dim B$ will denote the linear dimension of $B$; i.e., the power of a maximal set of linearly independent elements of $B$. Further, $[B]$ will denote the linear set spanned by the elements of $B$. For each $x$ in $A$, we shall denote by $A(x)$ the subalgebra generated by $x$. The algebra $A$ is called algebraic if $A(x)$ is finite dimensional for every $x$ in $A$. The algebra $A$ is said to be a division algebra if for every $a, b$ in $A$, with $a \neq 0$, the equations $ax = b$ and $ya = b$ are solvable in $A$.

An algebra over $R$ is called absolute-valued if it is a normed space under a multiplicative norm $| |$; i.e., a norm satisfying, in addition to the usual requirements, the condition $|xy| = |x| \cdot |y|$ for any $x, y$ in $A$. It is obvious that an absolute-valued algebra contains no divisors of zero.

A. A. Albert has shown [2, p. 768] that:

(*) An absolute-valued algebraic algebra with a unit element is isomorphic to either the real field $R$, the complex field $C$, the quaternion algebra $Q$, or the Cayley-Dickson algebra $D$.

F. B. Wright has proved [6, p. 332] the same theorem for absolute-valued division algebras with a unit element. In the present note we extend this result to an arbitrary absolute-valued algebra with a unit element.

First, we shall give a simple example of an infinite dimensional algebra which is absolute-valued. The existence of such an algebra shows that the assumption of the presence of a unit element is essential.

Let $A_0$ be the space of all sequences $x = \{x_n\}$ of real numbers with convergent series $\sum_{n=1}^{\infty} x_n^2$. $A_0$ is a Hilbert space over $R$ with respect to the norm $|x| = (\sum_{n=1}^{\infty} x_n^2)^{1/2}$, and with the usual addition and scalar multiplication: $\{x_n\} + \{y_n\} = \{x_n + y_n\}$, $\lambda \{x_n\} = \{\lambda x_n\}$. Let $\phi$...
be a one-to-one correspondence of the set of all pairs of natural numbers onto the set of all natural numbers. We define the multiplication of elements in $A_0$ as follows: $\{x_n\} \{y_n\} = \{z_n\}$, where $z_{\phi(m,k)} = x_m y_k \ (m, k = 1, 2, \cdots)$. This product makes $A_0$ an algebra over $R$. Moreover, $A_0$ is absolute-valued. Indeed, we have the equality

$$|xy| = \left( \sum_{n=1}^{\infty} z_n^2 \right)^{1/2} = \left( \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} z_{\phi(m,k)}^2 \right)^{1/2}$$

$$= \left( \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} x_m^2 y_k^2 \right)^{1/2} = \left( \sum_{m=1}^{\infty} x_m^2 \right)^{1/2} \left( \sum_{k=1}^{\infty} y_k^2 \right)^{1/2}$$

$$= |x| \cdot |y|.$$

Since $A_0$ is a Hilbert space, the function $|x|^2$ from $A_0$ to $R$ is a quadratic form admitting composition with respect to this multiplication. The complete structure theory of algebras (over any field) admitting such a form has been given by Kaplansky [4, p. 957], under the hypothesis that the algebra has a unit element. The algebra $A_0$ above shows that this hypothesis of existence of a unit element is essential to Kaplansky's results.

Throughout this paper, $A$ will denote an absolute-valued algebra.

**Lemma 1.** If all the elements of a subset $B$ of $A$ commute with each other, then $[B]$ is an inner-product space.

**Proof.** For every pair $x, y$ of elements of $B$ we have $(x+y)^2 - (x-y)^2 = 4xy$. Consequently, for $|x| = |y| = 1$, we get the inequality

$$|x+y|^2 + |x-y|^2 = |(x+y)^2| + |(x-y)^2| \geq 4|x| \cdot |y|.$$

Hence, according to Schoenberg's theorem [5, p. 962], we know that $[B]$ is an inner-product space over $R$.

**Lemma 2.** Let $x$ and $y$ be a pair of linearly independent elements of $A$. If $x$ commutes with $y$ and if $|x| = 1$, then there exists an element $y_0$ such that $[x, y_0] = [x, y]$ and

$$|\lambda x + \mu y_0|^2 = \lambda^2 + \mu^2$$

for any $\lambda, \mu$ in $R$.

**Proof.** By Lemma 1, $[x, y]$ is an inner-product space. Since $x$ and $y$ are linearly independent, $[x, y]$ is a two-dimensional linear space. There is then an element $y_0$ with $|y_0| = 1$ such that $y_0$ is orthogonal to $x$ and such that $[x, y] = [x, y_0]$. Equation (1) is a direct consequence of the orthogonality of $x$ and $y_0$.  

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**Lemma 3.** Let \( x, y \in A \), \( |x| = |y| = 1 \), and \( |x - y| = 2 \). If \( x \) commutes with \( y \), then \( x + y = 0 \).

**Proof.** If \( x = \lambda y \), \( \lambda \in \mathbb{R} \), we have the equalities \( |\lambda| = |\lambda y| = |x| = 1 \), \( |\lambda - 1| = |\lambda y - y| = |x - y| = 2 \), which imply \( \lambda = -1 \). Thus \( x + y = 0 \).

Suppose then that \( x \) and \( y \) are linearly independent and commute with each other. Then, by Lemma 2, there exists an element \( y_0 \) such that \( [x, y] = [x, y_0] \) and

\[
|x + \mu y_0|^2 = \lambda^2 + \mu^2.
\]

The element \( y \) can be written in the form

\[
y = \alpha x + \beta y_0,
\]

where, by (2), we have

\[
1 = |y|^2 = \alpha^2 + \beta^2.
\]

Furthermore, we have the equality

\[
4 = |x - y|^2 = |(1 - \alpha)x + \beta y_0|^2 = (1 - \alpha)^2 + \beta^2.
\]

From (4) we have \( \alpha = -1 \) and \( \beta = 0 \). Substituting in (3) yields \( x + y = 0 \).

**Theorem 1.** An absolute-valued algebra with a unit element is isomorphic to either the real field \( \mathbb{R} \), the complex field \( \mathbb{C} \), the quaternion algebra \( \mathbb{Q} \), or the Cayley-Dickson algebra \( \mathbb{D} \).

**Proof.** By virtue of Albert's theorem (\( * \)), it is sufficient to show that every absolute-valued algebra \( A \) with a unit element \( e \) is algebraic. We will show that

\[
x^2 \in [e, x]
\]

for any \( x \) in \( A \), which is equivalent to the inclusion \( A(x) \subseteq [e, x] \), and which consequently implies the inequality \( \dim A(x) \leq 2 \).

If \( e \) and \( x \) are linearly dependent, then (5) is obvious. Let us suppose that they are linearly independent. By Lemma 2, there exists an element \( x_0 \) such that

\[
[e, x] = [e, x_0],
\]

and such that

\[
|\lambda e + \mu x_0|^2 = \lambda^2 + \mu^2 \tag{\( \lambda, \mu \in \mathbb{R} \)).
\]

From (6) follows the relation

\[
x^2 \in [e, x_0, x_0^2].
\]
Further, from (7) we have the equality
\[ |e - x_0^2| = |(e - x_0)(e + x_0)| = |e - x_0| \cdot |e + x_0| = 2.\]
Since \(e\) commutes with \(x_0\) and since \(|e| = 1 = |x_0| = |x_0^2|\), Lemma 3 asserts that \(e+x_0^2=0\). Thus \([e, x_0, x_0^2] = [e, x_0]\). This, together with (6) and (8), gives relation (5). The theorem is thus proved.

The real field \(R\) is the unique one-dimensional absolute-valued algebra. The structure of two-dimensional absolute-valued algebras is also well-known. In particular, every two-dimensional commutative absolute-valued algebra is isomorphic to either the complex field \(C\) or to the algebra \(C^*\) of all complex numbers with the usual addition and scalar multiplication, where the product of \(x\) and \(y\) is equal to \(xy\) \([1; 3]\).

**Theorem 2.** If an absolute-valued algebra \(A\) contains an element \(a \neq 0\) which commutes with every element of \(A\) and which is alternative, i.e., which satisfies the equations
\[
(9) \quad a(ax) = a^2x, \quad (xa)a = xa^2,
\]
for any \(x\) in \(A\), then \(A\) has a unit element.

**Proof.** We may suppose, without loss of generality, that \(|a| = 1\). If \(a^2 = \lambda a\), where \(\lambda\) is in \(R\), we may set \(e = \lambda^{-1}a\), and we have
\[
e^2 = \lambda^{-2}a^2 = \lambda^{-1}a = e,
e(ex) = \lambda^{-2}a(ax) = \lambda^{-2}a^2x = e^2x = ex.
\]
Hence we have the equation
\[
e(ex - x) = e(ex) - ex = 0,
\]
which implies \(ex - x = 0\) for any \(x\) in \(A\). Since \(e\) commutes with every element of \(A\), \(e\) is a unit element of \(A\).

Now let us suppose that \(a\) and \(a^2\) are linearly independent. By Lemma 2, there exists an element \(b\) such that
\[
(10) \quad [a, a^2] = [a, b]
\]
and such that \(|\lambda a + \mu b|^2 = \lambda^2 + \mu^2\), for \(\lambda, \mu\) in \(R\). Hence we get
\[
(11) \quad |a^2 - b^2| = |(a - b)(a + b)| = |a - b| \cdot |a + b| = 2,
(12) \quad |b^2| = |b| = 1.
\]
Since \(a\) commutes with every \(x\) in \(A\), we have, according to (9), \(a^2x = xa^2\) for any \(x\) in \(A\). Therefore
\[
(13) \quad xy = yx \quad \text{for } x \in A, \quad y \text{ in } [a, a^2].
\]

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Taking into account (11), (12), and the relation $a^2b^2 = b^2a^2$, we have, by virtue of Lemma 3, $a^2 + b^2 = 0$. From (10) we get the representation $b = \alpha a + \beta a^2$, where $\beta \neq 0$ since $a$ and $a^2$ are linearly independent. Hence $a^2 + \alpha^2a^2 + 2\alpha\beta aa^2 + \beta^2(a^2)^2 = a^2 + b^2 = 0$. From (9) we have the equality

$$a((1 + \alpha^2)a + 2\alpha\beta a^2 + \beta^2aa^2) = a^2 + b^2 = 0,$$

which implies $(1 + \alpha^2)a + 2\alpha\beta a^2 + \beta^2aa^2 = 0$. Thus $aa^2$ is in $[a, a^2]$. Writing $aa^2 = \lambda a + \mu a^2$, we have, according to (9),

$$(a^2)^2 = a(aa^2) = \lambda a^2 + \mu aa^2 \in [a, a^2].$$

Hence it follows that the product of an arbitrary pair of elements of $[a, a^2]$ also belongs to $[a, a^2]$. In other words, $[a, a^2]$ is a subalgebra of $A$, and $A(a) = [a, a^2]$. Hence $A(a)$ is a two-dimensional commutative absolute-valued algebra. Since the algebra $C^*$ does not contain an element $a \neq 0$ satisfying (9), $A(a)$ is isomorphic to the complex field $C$. Then there are elements $e_0$, $i_0$ such that $A(a) = [e_0, i_0]$, $e_0^2 = e_0$, $i_0^2 = -e_0$, $e_0i_0 = i_0e_0 = i_0$.

To prove the theorem, it is sufficient to show that $A(a) = A$. Let us suppose to the contrary, that there exists an element $x$ not in $A(a)$. Since, according to (13), $e_0$ and $i_0$ commute with $x$, then $[e_0, i_0, x]$ is an inner product space. There is then an element $y$ in $[e_0, i_0, x]$, with $|y| = 1$, orthogonal to both $e_0$ and $i_0$. Consequently

$$|y^2 - e_0| = |(y - e_0)(y + e_0)| = |y - e_0| |y + e_0| = 2,$$

$$|y^2 + e_0| = |(y - i_0)(y + i_0)| = |y - i_0| |y + i_0| = 2.$$

By Lemma 3, $y^2 - e_0 = 0$ and $y^2 + e_0 = 0$, which implies the contradiction $e_0 = 0$. The theorem is thus proved.

**Theorem 3.** A commutative absolute-valued algebra is isomorphic either to the real field $R$, or to the complex field $C$, or to the algebra $C^*$.

**Proof.** It is sufficient to show that every commutative absolute-valued algebra is at most two-dimensional. Suppose $\dim A \geq 3$. By Lemma 1, $A$ is an inner-product space. There exist orthonormal elements $i_1$, $i_2$, $i_3$ in $A$. From the equalities

$$|i_3^2 - i_1^2| = |(i_3 - i_1)(i_3 + i_1)| = |i_3 - i_1| |i_3 + i_1| = 2,$$

$$|i_3^2 - i_2^2| = |(i_3 - i_2)(i_3 + i_2)| = |i_3 - i_2| |i_3 + i_2| = 2,$$

and from Lemma 3, it follows that $i_3^2 + i_1^2 = 0$ and $i_3^2 + i_2^2 = 0$. Hence $i_3^2 - i_2^2 = (i_1 + i_2)(i_1 - i_2) = 0$, which implies either $i_1 + i_2 = 0$ or $i_1 - i_2 = 0$. 

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Since $i_1$ and $i_2$ are orthonormal, either is a contradiction. Hence $\dim A \leq 2$.

**References**


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