A NOTE ON SEMI-GROUPS IN A LOCALLY COMPACT GROUP

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1. Introduction. In a recent paper by this author and others [1], the following theorem is proved:

Theorem 3 (Simon). In a compact group, every semi-group which contains a set of positive measure is an open and closed subgroup and therefore is itself measurable.

In this paper, we show that this result can be improved to the following:

Theorem A. In a locally compact group, every semi-group of nonzero inner measure and finite outer measure is an open compact subgroup.

This theorem can also be used to show an elusive¹ point in Theorem 5 of [1].

2. Proof of Theorem A. We rely heavily in this proof on Theorem 1 of [1], which states:

Theorem 1. Let G be a locally compact topological group with completed Haar measure μ and outer measure μ*. Let A, B ⊆ G be sets such that μ(A) > 0 and μ*(B) > 0. Then the interior of BA (also AB) is nonempty.

Let S now be a semi-group in a locally compact group G with Haar measure μ. We assume that μ*(S) > 0, so that S contains a measurable set of nonzero measure. Thus, by Theorem 1, the interior of $S^a \subset S$ is nonempty. Hence $S_0$, the interior of S, is also nonempty. Since S has finite outer measure, $S_0$ is measurable of finite measure. It is also clear that $S_0 \cdot S \subset S_0$. For each $s \in S_0$, we have

\[ s \cdot S_0 \subset S_0^a \subset S_0 \cdot S \subset S_0, \]

so that, since μ is left invariant,

\[ \mu(S_0) = \mu(s \cdot S_0) \leq \mu(S_0^a) \leq \mu(S_0). \]

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¹ The author expresses his debt to Mr. John E. Lange for pointing out this elusive-ness.

992
Therefore

\[ \mu(S_0 \setminus s \cdot S_0) = 0. \]

Since \( s \cdot S_0 \subseteq \overline{s \cdot S_0} \), we see that \( \mu(S_0 \setminus s \cdot S_0) = 0. \) But \( S_0 \setminus s \overline{S_0} \) is open, so that it must be empty, and \( s \cdot \overline{S_0} \supset S_0. \) We now set \( U = s^{-1} \cdot S_0. \) This is clearly a neighborhood of the identity in \( G \), and we have \( s \cdot \overline{S_0} \supset S_0 \supset S_0 = s \cdot U \cdot S_0 \supset S \cdot \overline{S_0}. \) Thus, \( S_0 = s \cdot \overline{S_0} \), and \( S_0 \) is closed, so that \( S_0 = s \cdot S_0. \)

It follows immediately that \( S_0 \) is an open and closed subgroup of \( G. \) We now have

\[ S = e \cdot S \subseteq S_0 \cdot S \subseteq S_0 \subseteq S. \]

Therefore \( S = S_0. \) Since \( S \) is an open and closed subgroup of finite measure, \( S \) is compact.

3. Proof of part of Theorem 5 in [1]. Part of Theorem 5 of [1] requires the proof that if in a locally compact group \( G \) we have two measurable sets \( T \) and \( S \) with \( T \) a subgroup and \( S \) a sub-semigroup of \( G, \) and if \( \mu(T) = \infty \) and \( \mu(T \setminus S) < \infty, \) then we have \( T \subseteq S. \) We construct the set \( A = T \cap (S \setminus S^{-1}); \) then we observe that \( A^{-1} \) is a semi-group, for if \( a_1, a_2 \in A, \) then \( a_2 a_1 \in S. \) However, \( a_1^{-1} a_2^{-1} \in S, \) for that would give us \( a_2^{-1} a_1 S \subseteq S, \) and \( a_2 \in S^{-1}, \) contrary to assertion. Thus, \( a_1^{-1} a_2^{-1} \in A^{-1}, \) which is thus a semi-group. Since \( A^{-1} = T \cap (S^{-1} \setminus S), \) we see that \( \mu(A^{-1}) \leq \mu(T \setminus S) < \infty. \) If \( \mu(A^{-1}) > 0, \) we have, by Theorem A, that \( A^{-1} \) is an open subgroup, and thus contains \( e, \) which is false, since \( A \cap A^{-1} = \emptyset. \) Thus \( \mu(A) = 0, \) and \( T \cap S \cap S^{-1}, \) which is a sub-group of \( T, \) has the additional property that

\[ \mu(T \cap S \cap S^{-1}) = \infty, \]

\[ \mu(T \setminus (S \cap S^{-1})) = \mu(T \setminus S) + \mu(A) < \infty. \]

It now follows easily (cf. [1, Lemma 5.1]) that \( T = (S \cap S^{-1}) \cap T, \) which gives us \( T \subseteq S. \)

Bibliography


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