A NOTE ON SEMI-GROUPS IN A LOCALLY COMPACT GROUP

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1. Introduction. In a recent paper by this author and others [1], the following theorem is proved:

Theorem 3 (Simon). In a compact group, every semi-group which contains a set of positive measure is an open and closed subgroup and therefore is itself measurable.

In this paper, we show that this result can be improved to the following:

Theorem A. In a locally compact group, every semi-group of nonzero inner measure and finite outer measure is an open compact subgroup.

This theorem can also be used to show an elusive\(^1\) point in Theorem 5 of [1].

2. Proof of Theorem A. We rely heavily in this proof on Theorem 1 of [1], which states:

Theorem 1. Let \( G \) be a locally compact topological group with completed Haar measure \( \mu \) and outer measure \( \mu^* \). Let \( A, B \subset G \) be sets such that \( \mu(A) > 0 \) and \( \mu^*(B) > 0 \). Then the interior of \( BA \) (also \( AB \)) is nonempty.

Let \( S \) now be a semi-group in a locally compact group \( G \) with Haar measure \( \mu \). We assume that \( \mu^*(S) > 0 \), so that \( S \) contains a measurable set of nonzero measure. Thus, by Theorem 1, the interior of \( S^2 \subset S \) is nonempty. Hence \( S_0 \), the interior of \( S \), is also nonempty. Since \( S \) has finite outer measure, \( S_0 \) is measurable of finite measure. It is also clear that \( S_0 \cdot S \subset S_0 \). For each \( s \in S_0 \), we have

\[
s \cdot S_0 \subset S_0^2 \subset S_0 \cdot S \subset S_0,
\]

so that, since \( \mu \) is left invariant,

\[
\mu(S_0) = \mu(s \cdot S_0) \leq \mu(S_0^2) \leq \mu(S_0).
\]

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\(^1\) The author expresses his debt to Mr. John E. Lange for pointing out this elusive-ness.

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Therefore
\[ \mu(S_0 \setminus s \cdot S_0) = 0. \]

Since \( s \cdot S_0 \subseteq s \cdot S_0 \), we see that \( \mu(S_0 \setminus s \cdot S_0) = 0 \). But \( S_0 \setminus S_0 \) is open, so that it must be empty, and \( s \cdot S_0 \supseteq S_0 \). We now set \( U = s^{-1} \cdot S_0 \). This is clearly a neighborhood of the identity in \( G \), and we have \( s \cdot S_0 \supseteq S_0 \supseteq S_0 \cdot S_0 = s \cdot U \cdot S_0 \supseteq s \cdot S_0 \). Thus, \( S_0 = s \cdot S_0 \), and \( S_0 \) is closed, so that \( S_0 = s \cdot S_0 \).

It follows immediately that \( S_0 \) is an open and closed subgroup of \( G \).

We now have
\[ S = e \cdot S \subseteq S_0 \cdot S \subseteq S_0 \subseteq S. \]

Therefore \( S = S_0 \). Since \( S \) is an open and closed subgroup of finite measure, \( S \) is compact.

3. Proof of part of Theorem 5 in [1]. Part of Theorem 5 of [1] requires the proof that if in a locally compact group \( G \) we have two measurable sets \( T \) and \( S \) with \( T \) a subgroup and \( S \) a sub-semigroup of \( G \), and if \( \mu(T) = \infty \) and \( \mu(T \setminus S) < \infty \), then we have \( T \subseteq S \). We construct the set \( A = T \cap (S \setminus S^{-1}) \); then we observe that \( A^{-1} \) is a semigroup, for if \( a_1, a_2 \in A \), then \( a_2a_1 \in S \). However, \( a_1^{-1}a_2^{-1} \in S \), for that would give us \( a_2^{-1}a_1 \in S \subseteq S \), and \( a_2 \in S^{-1} \), contrary to assertion. Thus, \( a_1^{-1}a_2^{-1} \in A^{-1} \), which is thus a semi-group. Since \( A^{-1} = T \cap (S^{-1} \setminus S) \), we see that \( \mu(A^{-1}) \leq \mu(T \setminus S) < \infty \). If \( \mu(A^{-1}) > 0 \), we have, by Theorem A, that \( A^{-1} \) is an open subgroup, and thus contains \( e \), which is false, since \( A \cap A^{-1} = \emptyset \). Thus \( \mu(A) = 0 \), and \( T \cap S \cap S^{-1} \), which is a sub-group of \( T \), has the additional property that
\[ \mu(T \cap S \cap S^{-1}) = \infty, \]
\[ \mu(T \setminus (S \cap S^{-1})) = \mu(T \setminus S) + \mu(A) < \infty. \]

It now follows easily (cf. [1, Lemma 5.1]) that \( T = (S \cap S^{-1}) \cap T \), which gives us \( T \subseteq S \).

Bibliography


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