SUMMABILITY OF A CLASS OF FOURIER SERIES

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1. In this section we shall consider a class of summability methods which sum the Fourier series

\[ \frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \]

of a function \( f(x) \) which is Lebesgue integrable and satisfies the condition

\[ |f(x + h) - f(x)| = o \left[ \left( \log \frac{1}{|h|} \right)^{-1} \right] \]

for some \( x \). It is known that the Fourier series of such functions need not converge [3, p. 173]. The partial sums, \( s_k(x) \), of (1) (since without loss of generality we may suppose \( f(x) \) to be even) are given by:

\[ s_k(x) = \frac{2}{\pi} \int_0^x f(x + t)D_k(t)dt, \]

where

\[ D_k(t) = \frac{\sin \left( k + \frac{1}{2} \right) t}{2 \sin \frac{1}{2} t}; \]

\( D_k(t) \) is called the Dirichlet kernel.

Suppose now that the matrix \( A = (a_{n,k}) \) determines a regular summability method. We shall also assume that \( A \) is a triangular matrix, i.e., \( a_{n,k} = 0 \) for \( k \geq n+1 \). The \( A \) transforms \( t_n(x) \) of the partial sums \( \{s_k(x)\} \) may then be written as follows:

\[ t_n(x) = \sum_{k=1}^{n} a_{n,k}s_k(x) = \frac{2}{\pi} \sum_{k=1}^{n} a_{n,k} \int_0^x f(x + t)D_k(t)dt. \]

Regular matrices which satisfy the condition

\[ \lim_{n \to \infty} \sum |a_{n,k} - a_{n,k+1}| = 0 \]

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are called strongly regular \([1; 2]\). We shall employ a slightly stronger condition in the theorem that follows. The proof of this theorem parallels that of a similar theorem due to Hardy and Littlewood \([3, \text{p. 34}]\).

**Theorem.** If

(i) \(A\) is a regular triangular matrix such that for some \(0 < r < 1\),

\[
\lim_{n \to \infty} \sum_{k=0}^{\infty} k^r \left| a_{n,k} - a_{n,k+1} \right| = 0,
\]

(ii) \(f(t)\) is Lebesgue integrable over \((0, \pi)\) and

\[
\left| f(x + h) - f(x) \right| = o \left[ \left( \log \frac{1}{h} \right)^{-1} \right]
\]

for some \(x\) then at the point \(x\), the Fourier series of \(f(t)\) is summable by \(A\) to \(f(x)\).

**Proof.** Clearly we may suppose that \(f(t)\) is an even function and \(f(x) = 0\). We then have

\[
\left| \sum_{k=1}^{n} a_{n,k} \int_{0}^{\pi} f(x + t) D_k(t) \, dt \right| \leq \left| \sum_{k=1}^{n} a_{n,k} \int_{0}^{\pi} f(x + t) D_k(t) \, dt \right| + \left| \sum_{k=1}^{n} a_{n,k} \int_{k-1}^{\pi} f(x + t) D_k(t) \, dt \right|
\]

(3)

\[
= \left| \sum_{k=1}^{n} a_{n,k} P_k(x) \right| + \left| \sum_{k=1}^{n} a_{n,k} Q_k(x) \right| + \left| \sum_{k=1}^{n} a_{n,k} R_k(x) \right|
\]

Since \(\left| D_k(t) \right| \leq M k\) in \((0, k^{-1})\)

\[
\left| P_k(x) \right| = \left| \int_{0}^{k^{-1}} f(x + t) D_k(t) \, dt \right| \leq M k \int_{0}^{k^{-1}} \left| f(x + t) \right| \, dt \to 0
\]

as \(k \to \infty\), since \(f(x+t)\) is continuous at \(x\). Consequently \(\{P_k(x)\}\) is a null sequence. Also \(\left| D_k(t) \right| \leq M' t\) and

\[
\left| Q_k(x) \right| = \left| \int_{k^{-1}}^{\pi} f(x + t) D_k(t) \, dt \right| \leq M o \left[ (\log k)^{-1} \right] \int_{k^{-1}}^{\pi} \frac{dt}{t}
\]

\[
= M' o \left[ (\log k)^{-1} \right] \left[ \log k - r \log k \right].
\]

Clearly \(Q_k(x)\) is a null sequence and the first two sums in (3) can be made as small as we may wish by choosing \(n\) sufficiently large. For some \(\delta\), \(\left| f(x+t) \right| < 1\) for \(0 \leq t \leq \delta\), and for large \(k\) such that \(k^{-r} > \delta\)
\[ R_k(x) = \int_{k^{-r}}^{\delta} f(x + t) D_k(t) dt + \int_{-\delta}^{k^{-r}} f(x + t) D_k(t) dt = R_k'(x) + R_k''(x). \]

By the Riemann-Lebesgue theorem, \( \{ R_k''(x) \} \) is a null sequence and so is \( A \) summable to zero. We must now evaluate \( |\sum a_{n,k} R_k'(x)| \).

First we introduce the notation
\[
\sum_{n=0}^{N} b_n(t) + \cdots + b_k(t) = \sum_{n=1}^{k} K_n(t),
\]
and observe that \( |\sum_{n=1}^{k} K_n(t)| \leq \frac{M''}{l^2}. \) For some \( k, k = 1, 2, \ldots, N, k^{-r} \geq \delta. \) However, since \( \delta \) (and hence \( N \)) is fixed,

\[
\lim_{n \to \infty} \sum_{n=1}^{N} |a_{n,k}| = 0.
\]

Consider the matrix \( (b_{n,k}), b_{n,k} = 0, k = 1, 2, \ldots, N, b_{n,k} = a_{n,k} \) elsewhere. The matrices \( (b_{n,k}) \) and \( (a_{n,k}) \) are equivalent, indeed for any sequence \( \{ s_k \} \)

\[
\lim_{n \to \infty} \sum_{n=1}^{N} (a_{n,k} - b_{n,k}) s_k = 0.
\]

Therefore in the sequel we shall assume without any loss of generality that \( a_{n,k} = 0, k = 1, 2, \ldots, N. \) We have for \( n > N \)

\[
\left| \sum_{k=1}^{n} a_{n,k} R_k'(x) \right| = \left| \sum_{k=N}^{n} a_{n,k} \int_{k^{-r}}^{\delta} f(x + t) [K_k(t) - K_{k-1}(t)] dt \right|
\leq \left| \sum_{k=N}^{n} (a_{n,k} - a_{n,k+1}) \int_{k^{-r}}^{\delta} f(x + t) K_k(t) dt \right| + \left| \sum_{k=N+1}^{n} a_{n,k} \int_{k^{-r}}^{(k-1)^{-r}} f(x + t) K_k(t) dt \right|.
\]

It then follows that

\[
\left| \sum_{k=N}^{n} (a_{n,k} - a_{n,k+1}) \int_{k^{-r}}^{\delta} f(x + t) K_k(t) dt \right|
\leq \sum_{k=N}^{n-1} |a_{n,k} - a_{n,k+1}| \int_{k^{-r}}^{\delta} \left| \frac{M''}{l^2} \right| dt
\leq M'' \sum_{k=N}^{n-1} (k^{-r} + \delta^{-1}) |a_{n,k} - a_{n,k+1}|
\]

and it is clear from the hypothesis
\begin{align*}
\lim_{n \to \infty} \sum_{k=N}^{n-1} (k^r + \delta^{-1}) | a_{n,k} - a_{n,k+1} | &= 0.
\end{align*}

Moreover,
\begin{align*}
\left| \sum_{k=N+1}^{n} a_{n,k} \int_{k}^{(k-1)-r} f(x + t)K_{k}(t)dt \right| &\leq \sum_{k=N+1}^{n} | a_{n,k} | \left| \int_{k}^{(k-1)-r} \frac{M''}{t^2} dt \right| \\
&= M'' \sum_{k=N+1}^{n} | a_{n,k} | [(k-1)^r - kr].
\end{align*}

Now the matrix \(| a_{n,k} |\) is not regular but does sum null sequences to zero if \((a_{n,k})\) is a regular matrix. The sequence \(\{(k-1)^r - kr\}\) is a null sequence for \(0 < r < 1\) and so
\begin{align*}
\lim_{n \to \infty} \sum_{k=1}^{n} | a_{n,k} | \left[ kr - (k + 1)^r \right] = 0
\end{align*}
and
\begin{align*}
limit_{n \to \infty} \left| \sum_{k=1}^{n} a_{n,k}R_{k}(x) \right| = 0.\end{align*}
This completes the proof of the theorem.

2. We have remarked before that there are functions of our class, even continuous functions, such that
\begin{align*}
\limsup_{k \to \infty} s_{k}(x) = \infty, \quad \text{(see [3, p. 173]).}
\end{align*}

Consider the matrix \(B = (b_{n,k})\) where \(b_{n,k} = 1/n\) for \(1 \leq k \leq n - 1\), \(b_{n,n} = 1/f(n)\), \(b_{n,k} = 0\) for \(k \geq n + 1\), and \(\lim_{n \to \infty} f(n) = \infty\). For any choice of \(f(n)\) so that \(\lim_{n \to \infty} b_{n,n} = 0\), \(B = (b_{n,k})\) is a strongly regular matrix. The Cesàro transform of a sequence is given by
\begin{align*}
l_{n} = \frac{s_{1} + \cdots + s_{n}}{n}.
\end{align*}

The \(B\) transform of a sequence is given by
\begin{align*}
\tau_{n} = \sum_{k=1}^{n} b_{n,k}s_{k} = \frac{n - 1}{n} l_{n-1}' + b_{n,n}s_{n}.
\end{align*}

It is well known that the Cesàro method sums the Fourier series of any Lebesgue integrable function almost everywhere [3, p. 45]. We can choose \(b_{n,n}\) so that \(\lim_{n \to \infty} b_{n,n} = 0\), but \(\limsup_{n \to \infty} b_{n,n}s_{n}(x) = \infty\) for the partial sums of Fourier series of functions of the class of our theorem. For such a choice \(\{\tau_{n}\}\) diverges, since \(\{(n-1)/n)l_{n-1}'\}\) converges while \(\{b_{n,n}s_{n}\}\) does not converge. In this way we see the class of matrices of our theorem cannot be extended to include all strongly regular matrices.
References


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