SOME CHARACTERIZATIONS OF RIEMANN $n$-SPHERES

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1. Introduction. The purpose of this paper is to extend a result of Feeman and Hsiung [1] characterizing the Riemann $n$-sphere. In addition, other similar characterizations are obtained.

Considerable use will be made of the following:

**Lemma.** Let $V^{n+1}$ be a Riemannian manifold of dimension $n+1 \geq 3$ and $V^n$ a closed orientable hypersurface of class $C^3$ imbedded in $V^{n+1}$. Then at each point of $V^n$ the mean curvatures $M_\alpha(M_0=1)$ satisfy

\[(1.1) \quad M_\alpha^2 - M_{\alpha-1}M_{\alpha+1} \geq 0 \]

for $\alpha=1, \cdots, n-1$; if in addition $M_s, M_{s-1}, \cdots, M_0$ are positive

\[(1.2) \quad M_{s-1}/M_s \geq M_{s-2}/M_{s-1} \geq \cdots \geq M_{s-t-1}/M_{s-t} \]

and if $M_s, M_{s-1}, \cdots, M_1$ are positive

\[(1.3) \quad M_1 \geq M_{2}^{1/2} \geq \cdots \geq M_{s}^{1/s}. \]

If any of the above is an equality at every point of $V^n$, then every point is an umbilic and $V^n$ is called a Riemann $n$-sphere.

2. $V^{n+1}$ of constant curvature. Throughout this section, $V^{n+1}$ will denote a Riemannian manifold of dimension $n+1 \geq 3$ and constant Riemannian curvature $K$ such that there is a normal coordinate system $S$ of Riemann at a fixed point $0$ covering the whole manifold. $V^n$ denotes a closed orientable hypersurface of class $C^3$ imbedded in $V^{n+1}$.

Under these conditions, Feeman and Hsiung [1] have shown that

\[(2.1) \quad \int_{V^n} M_{\alpha-1}dA + \int_{V^n} M_\alpha dA = 0 \]

for $\alpha=1, \cdots, n$, where $\rho$ is the scalar product of the unit normal vector $e_{n+1}$ of the hypersurface $V^n$ at the point $P$ and the position vector $Y$ of the point $P$ with respect to the coordinate system $S$.

**Theorem 1.** If there is an integer $s$, $1 \leq s \leq n$ such that $M_s > 0$ and either $\rho \leq -M_{s-1}/M_s$ or $\rho \geq -M_{s-1}/M_s$ throughout $V^n$, then $V^n$ is a Riemann $n$-sphere.

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PROOF. The cases $1 \leq s \leq n-1$ were proven by Feeman and Hsiung [1]. Therefore only the case $s = n$ will be considered here. Applying formula (2.1) with $\alpha = n$,

$$\int_{V^n} (M_{n-1} + M_n) dA = 0$$

and since $M_{n-1} + M_n$ is of fixed sign, $\rho = -M_{n-1}/M_n$. Now applying (2.1) with $\alpha = n-1$,

$$\int_{V^n} M_{n-2} dA = -\int_{V^n} M_{n-1} \rho dA,$$

$$= \int_{V^n} M_{n-1}^2/M_n dA,$$

or

$$\int_{V^n} (1/M_n)(M_{n-1}^2 - M_n M_{n-2}) dA = 0.$$

From (1.1), $(1/M_n)(M_{n-1}^2 - M_n M_{n-2}) \geq 0$ so that $M_{n-1}^2 - M_n M_{n-2} = 0$ at all points of $V^n$. By the lemma, $V^n$ is a Riemann $n$-sphere.

**Theorem 2.** If there are integers $s$ and $i$, $1 \leq i < s \leq n$, with $M_s, \ldots, M_i > 0$ and constants $c_i \geq 0$ for $i \leq j \leq s - 1$ such that at all points of $V^n$ one has $M_s = \sum c_j M_j$, then $V^n$ is a Riemann $n$-sphere.

**Proof.** By equation (1.2),

$$(M_j/M_s - M_{j-1}/M_{s-1}) = (M_j/M_{s-1})(M_{s-1}/M_s - M_{j-1}/M_j) \geq 0$$

for $i \leq j \leq s - 1$, and equality holds everywhere only if $V^n$ is a Riemann $n$-sphere. Thus

$$1 = \sum c_j(M_j/M_s) \geq \sum c_j(M_{j-1}/M_{s-1})$$

or

$$M_{s-1} - \sum c_j M_{j-1} \geq 0$$

with equality holding everywhere only if $V^n$ is a Riemann $n$-sphere. By (2.1),

$$\int_{V^n} (M_{s-1} - \sum c_j M_{j-1}) dA = -\int_{V^n} \rho \sum c_j M_j dA$$

$$= 0,$$

so that $M_{s-1} = \sum c_j M_{j-1}$ at all points of $V^n$. 

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Corollary. If there are integers \( s \) and \( i \), \( 1 \leq i < s \leq n \), with \( M_s, \ldots, M_i > 0 \) and a constant \( c \) with \( M_s = cM_s \) at all points of \( V^n \), then \( V^n \) is a Riemann \( n \)-sphere.

Theorem 3. If there are integers \( s \) and \( i \), \( 0 \leq i < s < n \), with \( M_{s+1}, \ldots, M_{i+1} > 0 \) and constants \( c_i \geq 0 \) for \( i \leq j \leq s - 1 \) such that at all points of \( V^n \), \( M_s = \sum c_j M_j \) and if \( p \) is of fixed sign throughout \( V^n \), then \( V^n \) is a Riemann \( n \)-sphere.

Proof. Applying the above procedure in reverse,
\[
M_{s+1} - \sum c_j M_{j+1} \leq 0
\]
and vanishing identically only if \( V^n \) is a Riemann \( n \)-sphere. By equation (2.1)
\[
\int_{\gamma^n} (M_{s+1} - \sum c_j M_{j+1}) p dA = -\int_{\gamma^n} (M_s - \sum c_j M_j) dA
\]
\[
= 0,
\]
and since \((M_{s+1} - \sum c_j M_{j+1}) p\) is of fixed sign, \( M_{s+1} = \sum c_j M_{j+1} \) everywhere.

Theorem 4. If there is an integer \( s \), \( 1 < s \leq n \), with \( M_s > 0 \) and a constant \( c \) with \( M_s = cM_{s-1} \) at all points of \( V^n \), then \( V^n \) is a Riemann \( n \)-sphere.

Proof. Since \( M_s > 0 \), \( c \) cannot be zero and \( M_{s-1} \) must be of fixed sign. By (1.1),
\[
M_{s-1}(M_{s-1} - cM_{s-2}) = M_{s-1}^2 - M_sM_{s-2} \geq 0,
\]
so that \( M_{s-1} - cM_{s-2} \) is of fixed sign and vanishes identically only if \( V^n \) is a Riemann \( n \)-sphere. Equation (2.1) gives
\[
\int_{\gamma^n} (M_{s-1} - cM_{s-2}) dA = \int_{\gamma^n} (cM_{s-1} - M_s) p dA
\]
\[
= 0,
\]
which implies \( M_{s-1} = cM_{s-2} \) at all points of \( V^n \).

Corollary. If \( M_n > 0 \) and the sum of the principal radii of curvature is a constant, then \( V^n \) is a Riemann \( n \)-sphere.

Proof. If \( r_i \) denotes the \( i \)th principal radius of curvature, then
\[
M_n = (n/\sum r_i)M_{n-1} = cM_{n-1}.
\]
This corollary is a special case of a theorem of Chern [3]. A result similar to the following has been obtained by Chern, Hano, and Hsiung [4] for the symmetric functions of the principal radii of curvature.

**Theorem 5.** If there is an integer $s$, $1 < s \leq n$, with $M_i > 0$ for $i = 1, \ldots, s$ and a constant $c$ with

$$
M_i^{1/(s-1)} \geq c \geq M_i^{1/s}
$$

at all points of $V^n$, and if $p$ is of fixed sign throughout $V^n$, then $V^n$ is a Riemann $n$-sphere.

**Proof.** Since $p$ is of fixed sign, (2.1) with $a = 1$ implies that $p < 0$. By (1.3), $M_1 \geq c$. Choose the orientation of $V^n$ for which $g^{} \equiv 0$ throughout $V^n$ implies $\int_{V^n} gdA \equiv 0$. Using (2.1),

$$
- \int_{V^n} c^{s-1} p M_1 dA \geq - \int_{V^n} c^s p dA
$$

$$
\equiv - \int_{V^n} M_s p dA
$$

$$
= \int_{V^n} M_{s-1} dA
$$

$$
\equiv \int_{V^n} c^{s-1} dA
$$

$$
= - \int_{V^n} c^{s-1} p M_1 dA.
$$

Hence all these terms are equal and

$$
\int_{V^n} p(M_1 - c) dA = 0
$$

so $M_1 = c$. By Theorem 3, with $s = 1$ and $i = 0$, $V^n$ is a Riemann $n$-sphere.

**Theorem 6.** If there is an integer $s$, $1 < s \leq n$, with $M_s, M_{s-1} > 0$ and a constant $c$ with

$$
M_{s-1}/M_s \geq c \geq M_{s-2}/M_{s-1}
$$

at every point of $V^n$, and if $p$ is of fixed sign throughout $V^n$, then $V^n$ is a Riemann $n$-sphere.

**Proof.** From (2.1) with $a = s$, one sees that $p < 0$. Choose the orientation of $V^n$ for which $g \geq 0$ throughout $V^n$ implies $\int_{V^n} gdA \geq 0$. From (2.1),

$$
\int_{V^n} p(M_1 - c) dA = 0
$$

so $M_1 = c$. By Theorem 3, with $s = 1$ and $i = 0$, $V^n$ is a Riemann $n$-sphere.
\[
\int_{V^n} M_{s-2} dA = - \int_{V^n} \rho M_{s-1} dA \\
= - \int_{V^n} \rho c M_s dA \\
= \int_{V^n} c M_{s-1} dA \\
\leq \int_{V^n} M_{s-2} dA.
\]

Hence, all these terms are equal and so

\[
\int_{V^n} \rho (M_{s-1} - c M_s) dA = 0,
\]

and \(\rho (M_{s-1} - c M_s) \leq 0\), which implies \(M_{s-1} = c M_s\) at every point of \(V^n\). By Theorem 4, \(V^n\) is therefore a Riemann \(n\)-sphere.

**Corollary.** If \(S\) is a closed orientable surface of class \(C^2\) twice differentiably imbedded in Euclidean 3-space and having \(M_2 > 0\), then either \(\inf_{x \in S} M_1^2 < \sup_{x \in S} M_2\) or \(S\) is a sphere.

**Proof.** By Hadamard's theorem, \(S\) is the boundary of a convex body, and choosing the origin in the interior of this body, \(\rho\) is of fixed sign. If \(\inf M_1^2 \geq \sup M_2\), there is a constant \(c > 0\) with \(M_1^2 \geq c^2 \geq M_2\). Since \(M_1\) is continuous, either \(M_1 \geq c\) or \(M_1 \leq -c\). In the first case, by Theorem 5, every point of \(S\) is an umbilic and so \(S\) is a sphere [2, p. 128]. In the second case, \(\rho\) must be positive and choosing the orientation as before,

\[
- \int_S c \rho M_1 dA = \int_S c dA \\
\leq - \int_S M_1 dA \\
= \int_S \rho M_2 dA \\
\leq \int_S \rho c^2 dA \\
\leq - \int_S c \rho M_1 dA.
\]
Thus all these terms are equal and one has $M_2 = -cM_1$. By Theorem 4, every point is an umbilic and $S$ is a sphere.

3. $n+1 \geq 4$. In this section $V^{n+1}$ will denote a Riemannian manifold of dimension $n+1 \geq 4$ and such that there is a normal coordinate system of Riemann at a fixed point 0 covering the whole manifold $V^{n+1}$. $V^n$ is a closed orientable hypersurface of class $C^3$ imbedded in $V^{n+1}$.

With these assumptions, Feeman and Hsiung [1] have proven that

$$\int_{V^n} M_\alpha d\Omega + \int_{V^n} M_\alpha p d\Omega = 0$$

for $\alpha$ odd and $1 \leq \alpha \leq n$.

**Theorem 7.** If there is an even integer $s$, $1 < s \leq n$, with $M_s$, $M_{s-1} > 0$ and $p \leq -M_{s-1}/M_s$ at all points of $V^n$, then $V^n$ is a Riemann $n$-sphere.

**Proof.** By (1.2), $p \leq -M_{s-1}/M_s \leq -M_{s-2}/M_{s-1}$ and by (3.1),

$$\int_{V^n} (M_{s-2} + pM_{s-1})d\Omega = 0,$$

so $p \leq -M_{s-1}/M_s \leq -M_{s-2}/M_{s-1} = p$. Thus $M_{s-1}/M_s = M_{s-2}/M_{s-1}$ at all points of $V^n$, and by the lemma $V^n$ is a Riemann $n$-sphere.

**Theorem 8.** If there is an even integer $s$, $1 < s < n$, with $M_s$, $M_{s+1} > 0$ and $p \geq -M_{s-1}/M_s$ at all points of $V^n$, then $V^n$ is a Riemann $n$-sphere.

**Proof.** By (1.2), $p \geq -M_{s-1}/M_s \geq -M_s/M_{s+1}$, and by (3.1),

$$\int_{V^n} (M_s + pM_{s+1})d\Omega = 0,$$

so $p \geq -M_{s-1}/M_s \geq -M_s/M_{s+1} = p$. Therefore $M_{s-1}/M_s = M_s/M_{s+1}$ at all points of $V^n$, and $V^n$ is a Riemann $n$-sphere.

For $s$ odd, the corresponding results were obtained by Feeman and Hsiung [1].

Applying formula (3.1) to the integrations in Theorems 2 and 3 one obtains:

**Theorem 9.** If there are odd integers $s$ and $i$, $1 \leq i < s \leq n$, with $M_s$, $\cdots$, $M_i > 0$ and constants $c_j \geq 0$ ($c_i = 0$ if $j$ is even) for $i \leq j \leq s-2$ such that at all points of $V^n$ one has $M_s = \sum c_j M_j$, then $V^n$ is a Riemann $n$-sphere.

**Theorem 10.** If there are even integers $s$ and $i$, $0 \leq i < s < n$, with
$M_{i+1}, \ldots, M_{s+1}>0$ and constants $c_i \geq 0$ ($c_i = 0$ if $j$ is odd) for $i \leq j \leq s-2$ such that $M_s = \sum c_j M_j$ at every point of $V^n$, and if $p$ is of fixed sign throughout $V^n$, then $V^n$ is a Riemann $n$-sphere.

**Theorem 11.** If there is an odd integer $s$, $1 < s \leq n$, with $M_i > 0$ for $i=1, \ldots, s$ and a constant $c$ with $M_i^{1/(n-1)} \geq c \geq M_1^{1/n}$ at all points of $V^n$, and if $p$ is of fixed sign throughout $V^n$, then $V^n$ is a Riemann $n$-sphere.

**Proof.** The inequalities in the proof of Theorem 5 still hold, since by (3.1) the integrations can be performed. Thus

$$\int_{V^n} p(M_s - c^{r-1} M_1) dA = 0.$$ 

One has $c^{r-1} M_1 \geq c \geq M_s$ and $p$ of fixed sign, so $M_s = c^{r-1} M_1$ everywhere. By Theorem 9, with $i=1$, $c_1 = c^{r-1}$ and $c_j = 0$ if $j \neq 1$, $V^n$ is a Riemann $n$-sphere.

**References**


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