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ISOMETRIES OF GROUP ALGEBRAS

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Let \(G\) be a locally compact abelian group, \(\hat{G}\) its character group, and \(A\) the group algebra of \(G\). Associated with any automorphism \(\phi\) of \(A\) \([2]\) is a homeomorphism \(\tau\) of \(\hat{G}\) onto itself, with the property that, for \(\alpha \in \hat{G}\), \(f \in A\) and \(\mathcal{F}\) the Fourier transform, \(\mathcal{F}(\phi(f))(\tau\alpha) = \mathcal{F}(f)(\alpha)\). Results of Helson \([2]\) and Wendel \([4]\) state that if \(e\) is the unit of \(\hat{G}\), then

\[
\tau(e) \tau(xy) = \tau(x) \tau(y), \quad \text{for all } x, y \in \hat{G},
\]

if and only if \(\phi\) is an isometry. The object of the present note is to give a further equivalent form of the statement that \(\phi\) is an isometry.

Let \(T_\alpha, \alpha \in \hat{G}\), be that operator on \(A\) which for all \(x \in G, f \in A\) satisfies \((T_\alpha f)(x) = f(x)(x, \alpha)\). We consider homomorphisms \(\phi\) of \(A\) onto \(A\) such that to each \(\alpha\) there is a \(\rho(\alpha) \in \hat{G}\) such that

\[
\phi T_\alpha = T_{\rho(\alpha)} \phi, \quad \alpha \in \hat{G}.
\]

Our result is that such homomorphisms are isomorphisms and indeed isometries.

**Theorem 1.** Let \(\phi\) be a homomorphism of the group algebra \(A\) of the locally compact abelian group \(G\) onto itself. Suppose that \(\phi\) satisfies (2); then \(\phi\) is an isomorphism.

Let \(K\) be the kernel of \(\phi\). Since \(\phi\) is automatically continuous \([3]\), \(K\) is a proper closed ideal of \(A\). The Wiener Tauberian theorem thus yields a maximal regular ideal \(M\) containing \(K\). From condition (2) it follows that for \(k \in K\) and \(\beta \in \hat{G}\), \(\phi T_\beta(k) = T_{\rho(\beta)} \phi(k) = 0\), and there-

Received by the editors September 11, 1959.

\(^1\) This research was sponsored in part by the National Science Foundation under grant NSF-G5865.
fore $T_{\alpha}(K) \subset K \subset M$. Let $k \in K$ and let $\alpha$ be the character corresponding to $M$; then $0 = \xi(T_{\beta}(k))(\alpha) = \xi(k)(\beta \alpha)$ for arbitrary $\beta \in \hat{G}$. Consequently $k = 0$, and $\phi$ is an isomorphism.

**Lemma.** Let $\phi$ be an automorphism of $A$ which satisfies (2), and let $\tau$ be the associated homeomorphism of $\hat{G}$ onto itself. Then:

(i) The mapping $\rho$ is a homomorphism of $\hat{G}$ onto itself.

(ii) For any $\alpha, \beta \in \hat{G}$, $\tau(\alpha \beta) = \rho(\beta) \tau(\alpha)$.

We show that $\rho$ is a homomorphism into $\hat{G}$. That the image of $\hat{G}$ under $\rho$ is all of $\hat{G}$ is then a consequence of (ii). For all $f \in A$, $T_{\rho(\alpha \beta)}\phi(f) = \phi T_{\alpha \beta}(f) = \phi T_{\alpha}(T_{\beta}(f)) = T_{\rho(\alpha)} T_{\rho(\beta)} \phi(f)$. Since $\phi(f)$ ranges over $A$, $T_{\rho(\alpha \beta)} = T_{\rho(\alpha)} T_{\rho(\beta)}$, whence $\rho(\alpha \beta) = \rho(\alpha) \rho(\beta)$.

It follows from (2) that for all $\beta \in \hat{G}$, $\xi(T_{\alpha}(f))(\tau(\beta)) = \xi(T_{\rho(\alpha)} \phi(f))(\tau(\beta))$. Thus, $\xi(\phi(f))(\rho(\alpha) \tau(\beta)) = \xi(T_{\rho(\alpha)} \phi(f))(\tau(\beta)) = \xi(\phi(f))(\tau(\beta)) = \xi(\phi(f))(\tau(\alpha \beta))$. Since the Fourier transforms of elements in $A$ separate the points of $\hat{G}$, we deduce that $\rho(\alpha) \tau(\beta) = \tau(\alpha \beta)$.

**Theorem 2.** Let $A$ be the group algebra of a locally compact abelian group $G$, and let $\phi$ be an automorphism of $A$. The following conditions are equivalent:

(a) $\phi$ is an isometry.

(b) The associated homeomorphism $\tau$ of $\hat{G}$ onto itself satisfies (1).

(c) $\phi$ satisfies (2).

In view of our earlier remarks it suffices to show that (b) is equivalent to (c). Suppose that $\phi$ satisfies (2). Let $\tau$ be the homeomorphism of $\hat{G}$ associated with $\phi$. From (ii) of the lemma, with the identity $e$ of $\hat{G}$ in the role of $\alpha$, we obtain $\tau(\beta) = \rho(\beta) \tau(e)$ for all $\beta \in \hat{G}$. Thus, using the above and (i) of the lemma,

$$\tau(e) \tau(\alpha \beta) = (\tau(e))^2 \rho(\alpha \beta) = \tau(e) \rho(e) \tau(e) \rho(\beta) = \tau(\alpha) \tau(\beta).$$

Hence (c) implies (b).

Suppose that $\tau$ satisfies (1). Recall that

$$\xi(\phi(f))(\tau(\alpha)) = \xi(f)(\alpha), \quad \alpha \in \hat{G}, f \in A. \quad (3)$$

By repeated application of (3) and the definition of $T_{\alpha}$, we obtain $\xi(\phi(T_{\alpha}(f))(\tau(\alpha))) = \xi(T_{\alpha}(f))(\alpha) = \xi(f)(\alpha \alpha) = \xi(\phi(f))(\tau(\alpha \alpha))$. If we now apply (1), the last term above becomes $\xi(\phi(f))(\tau(\alpha) \tau(\alpha \alpha) \tau(e)) \xi(\phi(f))(\tau(\alpha \alpha))^{-1} = \xi(T_{\tau(\alpha) \tau(\alpha \alpha) \tau(e)}(\alpha))(\tau(e))^2 \phi(f))$. If we compare the initial and terminal points of the string of equalities, and note that the range of $\tau$ is $\hat{G}$, we see that

$$\xi(\phi(T_{\alpha}(f))) = \xi(T_{\tau(\alpha) \tau(\alpha \alpha) \tau(e)}(\alpha))(\tau(e))^2 \phi(f)).$$
Thus $\phi T_\alpha(f) = T_{\tau(\alpha)(\tau(e))}^{-1}\phi(f)$. However $f$ is arbitrary in $A$, so $\phi T_\alpha = T_{\tau(\alpha)(\tau(e))}^{-1}$ and thus we may take $\rho(\alpha) = \tau(\alpha)(\tau(e))^{-1}$. Hence $\phi$ satisfies (2), and (b) implies (c).

If the mapping $\phi$ is an involution on $A$, a result of Civin and Yood [1] yields a homeomorphism $\sigma$ of period two of $\hat{G}$ onto itself such that $\hat{\phi}(f)(\sigma\alpha) = [\hat{\phi}(f)(\alpha)]^{-1}$. Methods totally similar to the above and to those of [2; 4] allow one to establish the following theorem.

**Theorem 3.** Let $A$ be the group algebra of a locally compact abelian group, and let $\phi$ be an involution on $A$. The following conditions are equivalent.

(a) $\phi$ is an isometry.

(b) The associated homeomorphism $\sigma$ of period two of $\hat{G}$ onto itself satisfies $\sigma(e)\sigma(xy) = \sigma(x)\sigma(y)$, for all $x, y \in \hat{G}$.

(c) $\phi$ satisfies (2).

**References**


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