Let $G$ be a locally compact abelian group, $\hat{G}$ its character group, and $A$ the group algebra of $G$. Associated with any automorphism $\phi$ of $A$ [2] is a homeomorphism $\tau$ of $\hat{G}$ onto itself, with the property that, for $\alpha \in \hat{G}$, $f \in A$ and $\hat{F}$ the Fourier transform, $\hat{F}(\phi(f))(\tau\alpha) = \hat{F}(f)(\alpha)$. Results of Helson [2] and Wendel [4] state that if $e$ is the unit of $\hat{G}$, then

$$\tau(e)\tau(xy) = \tau(x)\tau(y), \quad \text{for all } x, y \in \hat{G},$$

if and only if $\phi$ is an isometry. The object of the present note is to give a further equivalent form of the statement that $\phi$ is an isometry.

Let $T_\alpha, \alpha \in \hat{G}$, be that operator on $A$ which for all $x \in G, f \in A$ satisfies $(T_\alpha f)(x) = f(x)(x, \alpha)$. We consider homomorphisms $\phi$ of $A$ onto $A$ such that to each $\alpha$ there is a $\rho(\alpha) \in \hat{G}$ such that

$$\phi T_\alpha = T_{\rho(\alpha)} \phi, \quad \alpha \in \hat{G}.$$

Our result is that such homomorphisms are isomorphisms and indeed isometries.

**Theorem 1.** Let $\phi$ be a homomorphism of the group algebra $A$ of the locally compact abelian group $G$ onto itself. Suppose that $\phi$ satisfies (2); then $\phi$ is an isomorphism.

Let $K$ be the kernel of $\phi$. Since $\phi$ is automatically continuous [3], $K$ is a proper closed ideal of $A$. The Wiener Tauberian theorem thus yields a maximal regular ideal $M$ containing $K$. From condition (2) it follows that for $k \in K$ and $\beta \in \hat{G}$, $\phi T_{\beta}(k) = T_{\rho(\beta)} \phi (k) = 0$, and there-

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fore $T_\beta(K) \subset K \subset M$. Let $k \in K$ and let $\alpha$ be the character corresponding to $M$; then $0 = \mathcal{F}(T_\beta(k))(\alpha) = \mathcal{F}(k)(\beta \alpha)$ for arbitrary $\beta \in \hat{G}$. Consequently $k = 0$, and $\phi$ is an isomorphism.

**Lemma.** Let $\phi$ be an automorphism of $A$ which satisfies (2), and let $\tau$ be the associated homeomorphism of $\hat{G}$ onto itself. Then:

(i) The mapping $\rho$ is a homomorphism of $\hat{G}$ onto itself.

(ii) For any $\alpha, \beta \in \hat{G}$, $\tau(\alpha \beta) = \rho(\beta)\tau(\alpha)$.

We show that $\rho$ is a homomorphism into $\hat{G}$. That the image of $\hat{G}$ under $\rho$ is all of $\hat{G}$ is then a consequence of (ii). For all $f \in A$, $T_{\rho(\alpha \beta)}(f) = \phi T_{\alpha \beta}(f) = \phi T_\alpha T_\beta(f) = T_{\rho(\alpha)} T_{\rho(\beta)}(f)$. Since $\phi(f)$ ranges over $A$, $T_{\rho(\alpha \beta)}(f) = T_{\rho(\alpha)} T_{\rho(\beta)}(f)$, whence $\rho(\alpha \beta) = \rho(\alpha)\rho(\beta)$.

It follows from (2) that for all $\beta \in \hat{G}$,

\[ \mathcal{F}(T_{\alpha \beta}(f))(\tau(\beta)) = \mathcal{F}(T_{\alpha \beta}(f))(\tau(\beta)) = \mathcal{F}(T_{\alpha}(f))(\beta) = \mathcal{F}(f)(\alpha \beta) = \mathcal{F}(f)(\tau(\alpha \beta)). \]

Since the Fourier transforms of elements in $A$ separate the points of $\hat{G}$, we deduce that $\rho(\alpha)\tau(\beta) = \tau(\alpha \beta)$.

**Theorem 2.** Let $A$ be the group algebra of a locally compact abelian group $G$, and let $\phi$ be an automorphism of $A$. The following conditions are equivalent:

(a) $\phi$ is an isometry.

(b) The associated homeomorphism $\tau$ of $\hat{G}$ onto itself satisfies (1).

(c) $\phi$ satisfies (2).

In view of our earlier remarks it suffices to show that (b) is equivalent to (c). Suppose that $\phi$ satisfies (2). Let $\tau$ be the homeomorphism of $\hat{G}$ associated with $\phi$. From (ii) of the lemma, with the identity $e$ of $\hat{G}$ in the role of $\alpha$, we obtain $\tau(\beta) = \rho(\beta)\tau(e)$ for all $\beta \in \hat{G}$. Thus, using the above and (i) of the lemma,

\[ \tau(e)\tau(\alpha \beta) = (\tau(e))^2 \rho(\alpha \beta) = \tau(e)\rho(\alpha)\tau(e)\rho(\beta) = \tau(\alpha)\tau(\beta). \]

Hence (c) implies (b).

Suppose that $\tau$ satisfies (1). Recall that

\[ \mathcal{F}(f)(\tau(\alpha)) = \mathcal{F}(f)(\alpha), \quad \alpha \in \hat{G}, f \in A. \]

By repeated application of (3) and the definition of $T_\alpha$, we obtain

\[ \mathcal{F}(\phi T_\alpha(f))(\tau(\alpha \beta)) = \mathcal{F}(T_\alpha(f))(\tau(\alpha \beta)) = \mathcal{F}(\phi T_\alpha(f))(\tau(\alpha \beta)). \]

If we now apply (1), the last term above becomes

\[ \mathcal{F}(\phi(f))(\tau(\alpha \beta))(\tau(e))^{-1} = \mathcal{F}(T_{\tau(\alpha \beta)}(\tau(e)))^{-1}(\tau(\alpha \beta)) = \mathcal{F}(\phi(f))(\tau(\alpha \beta)). \]

If we compare the initial and terminal points of the string of equalities, and note that the range of $\tau$ is $\hat{G}$, we see that

\[ \mathcal{F}(\phi T_\alpha(f)) = \mathcal{F}(T_{\tau(\alpha \beta)}(\tau(e)))^{-1}(\tau(\alpha \beta)). \]
Thus \( \phi T_a(f) = T_{\tau(\rho)}(\tau(e))^{-1} \phi(f) \). However \( f \) is arbitrary in \( A \), so \( \phi T_a = T_{\tau(\rho)}(\tau(e))^{-1} \) and thus we may take \( \rho(\alpha) = \tau(\alpha)(\tau(e))^{-1} \). Hence \( \phi \) satisfies (2), and (b) implies (c).

If the mapping \( \phi \) is an involution on \( A \), a result of Civin and Yood [1] yields a homeomorphism \( \sigma \) of period two of \( \hat{G} \) onto itself such that \( \mathfrak{F}(\phi(f))(\sigma \alpha) = [\mathfrak{F}(f)(\alpha)]^{-1} \). Methods totally similar to the above and to those of [2; 4] allow one to establish the following theorem.

**Theorem 3.** Let \( A \) be the group algebra of a locally compact abelian group, and let \( \phi \) be an involution on \( A \). The following conditions are equivalent.

(a) \( \phi \) is an isometry.

(b) The associated homeomorphism \( \sigma \) of period two of \( \hat{G} \) onto itself satisfies \( \sigma(e)\sigma(xy) = \sigma(x)\sigma(y) \), for all \( x, y \in \hat{G} \).

(c) \( \phi \) satisfies (2).

**References**


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