AN EXAMPLE CONCERNING AFFINE CONNEXIONS

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Introduction. The purpose of this note is to exhibit a simply connected manifold on which is defined a complete affine connexion with the following property: for each integer \( n > 0 \) there are two points such that any broken geodesic between them must contain at least \( n \) breaks. The idea is to take a manifold, say \( M \), with complete affine connexion such that the exponential map is not onto, remove neighborhoods of two points, getting \( M' \), so that any geodesic (in \( M \)) leaving one of these neighborhoods must break at least once in order to intersect the other neighborhood, then attach a countable number of copies of \( M' \) together through these neighborhoods. The difficulty arises in accomplishing the attaching while preserving completeness. Perhaps this process may prove useful in other studies. This example shows that Theorem 1 in [3] (see bibliography) does not hold if a bound is placed on the number of breaks a broken geodesic may have.

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1. Local flattening of a connexion. Let \( M \) be a \( C^\infty \) manifold on which is defined a complete affine connexion \( \omega \), i.e., a connexion on the bundle of bases \( B(M) \) (see [1]). Take a point \( m \) in \( M \) and let \( \phi \) be a coordinate map with coordinate functions \( x_1, \ldots, x_n \) on a neighborhood \( U \) of \( m \). We take \( U \) so that the exponential map (for \( \omega \)) at \( m \) is a diffeo with respect to \( U \) and \( U \) is convex (see [4]). Let \( \Gamma' \) denote the set of functions \( \Gamma'_{ij} \) belonging to \( \phi \); we assume \( \phi \) is chosen so that all \( \Gamma'_{ij}(m) = 0 \). Further we suppose all \( x_i(m) = 0 \) and \( p \) in \( U \) implies \( \sum x_i(p)^2 < R^2 \). Let \( d \) denote the distance function in \( U \) induced by this coordinate system. For \( 0 < r < R \), let \( U(r) \) be the open ball about \( m \) of radius \( r \). In general, let \( B(\rho, r) = \{ q \in U: d(q, \rho) < r \} \).

**Theorem 1.** There is a complete connexion \( \hat{\omega} \) on \( M \) and a real number \( s \) such that: (1) \( 0 < s < R \), (2) \( \omega = \hat{\omega} \) on \( M - U(s) \), (3) \( \hat{\omega} \) is the flat Riemannian connexion associated with \( d \) in \( U(s/2) \).

**Proof.** For each \( r, 0 < r < R \), let \( f_r \) be a real valued \( C^\infty \) function on \( M \) with \( f_r(p) = 0 \) for \( p \) in \( U(r/2) \), \( f_r(p) = 1 \) for \( p \) in \( M - U(r) \), and \( 0 \leq f_r \leq 1 \). Moreover we may assume the mapping: \( (r, p) \rightarrow f_r(p) \) is continuous on the set \( (0, R) \times M \). Let \( f_0 \equiv 1 \) on \( M \). For each \( r, 0 \leq r < R \),

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let \( \omega_r \) be the connexion on \( M \) defined by requiring \( \omega_r = \omega \) on \( M - U \), while \( \Gamma_r = f_\Gamma \) defines \( \omega_r \) in terms of the coordinate map \( \phi \) in \( U \).

Note the function: \( (r, \phi) \to f_{ir}(\phi) \Gamma(\phi) \) is continuous on \(-R, R \times U\). This is trivial at points where \( r \neq 0 \). For a point \((0, \phi)\) take any \( \varepsilon > 0 \). If \( p = m \), let \( \delta > 0 \) be chosen so that \( q \in U(\delta) \) implies \(|\Gamma^i_m(q)| < \varepsilon \) for all \( i, j, k \). Then if \((r, q) \in (-R, R) \times U(\delta) \), \(|f_{ir}(q)\Gamma(q) - \Gamma(\phi)| < \varepsilon \).

If \( p \neq m \), let \( \delta_1 = \frac{d(m, \phi)}{2} \). Let \( \delta_2 > 0 \) be chosen so that \( q \in B(\phi, \delta_2) \) implies \(|\Gamma^i_m(q) - \Gamma^i_m(\phi)| < \varepsilon \) for all \( i, j, k \), and \( \delta_2 < \delta_1 \). Then if \((r, q) \in (-\delta_1, \delta_1) \times B(\phi, \delta_2) \), \(|f_{ir}(q)\Gamma(q) - \Gamma(\phi)| = |\Gamma(q) - \Gamma(\phi)| < \varepsilon \) since \( q \in M - U(\delta_1) \).

Let \( T(M) \) be the tangent bundle to \( M \); let \( \pi: T(M) \to M \); let \( W = \{(p, X) \in T(M): p \in U(R/2) \) and \( \exp_m(\pm X) \in U(R/2)\} \); let \( y_i = x_i \circ \pi \) and \( y_{n+i} = dx_i \) for \( i = 1, \ldots, n \). Then the differential equations for "lifted" geodesics for the \( \omega_r \) connexion become (the prime denote differentiation with respect to \( t) \),

\[
(y_i)' = y_{n+i}, \quad (y_{n+i})' = -\sum_{jk} (f_{ir} \Gamma^i_{jk} \circ \pi) y_{n+j} y_{n+k}
\]

summing \( j \) and \( k \) from 1 to \( n \). We abbreviate this system by writing \( y' = F(y, r) \). Note \( F \) is a continuous function on \( D = W \times (\frac{-R}{2}, \frac{R}{2}) \) and satisfies a Lipschitz condition in \( y \) uniformly on \( D \).

Use the notation \( g(t, (p, X), r) \) to denote a geodesic of the \( \omega_r \) connexion defined on some interval about \( t = 0 \), with \( 0 \to p \) and tangent vector \( X \) at \( t = 0 \). We know \( g(t, (p, X), 0) \) is defined for all \( t \) by completeness of \( \omega_r \), but the image points may not all remain in \( U \).

For each \((p, X)\) in \( W \) we know \( g(t, (p, X), 0) \) is defined and stays in \( U \) for \( t \) in \([-1, 1]\). A slight variation of a theorem from the theory of differential equations (see [2, p. 29]) gives us a \( \delta_1 > 0 \) such that if \(|r| < \delta_1 \) then \( g(t, (p, X), r) \) is defined for \( t \) in \([-1, 1]\). Now let

\[
W' = \left\{(p, X) \in W: \frac{3R}{8} \leq d\left(m, \exp_p \frac{1}{2} X \right) \leq \frac{R}{2}\right\}.
\]

Let \( V = U(R/8) \), let \( W'' = W' \cap \pi^{-1}(\overline{V}) \). Hence \( W'' \) is compact. The theorem from differential equations assures us \( g(t, (p, X), r) \) is continuous on the set \((-1, 1) \times W \times (-\delta_1, \delta_1)\) and hence is uniformly continuous on \( A = [-1/2, 1/2] \times W'' \times [-\delta_1/2, \delta_1/2] \). Thus for the number \( R/16 \) we obtain \( \delta_2 > 0 \) such that if we have elements \((t_1, (p_1, X_1), r_1)\) and \((t_2, (p_2, X_2), r_2)\), both in \( A \), with \(|t_1 - t_2| < \delta_2\), \( d((p_1, X_1), (p_2, X_2)) < \delta_2 \) and \(|r_1 - r_2| < \delta_2 \), then

\[
d(g(t_1, (p_1, X_1), r_1), g(t_2, (p_2, X_2), r_2)) < \frac{R}{16}.
\]
Now take $s$ such that $0 < s < \min\{\delta_1/2, \delta_2, R/8\}$. Claim $\bar{\omega} = \omega_1$ is the desired connexion. We need only show $\bar{\omega}$ is complete.

Since $\bar{\omega} = \omega$ in $M - U(s)$ we need only worry about geodesics emanating from (or passing through) $U(s)$. Take $p$ in $U(s)$ and take any $Y$ tangent to $M$ at $p$. There is a real number $b$ such that $(p, bY) \in W''$, i.e., $3R/8 \leq d(m, \exp_p \frac{1}{2} bY) \leq R/2$, for $U$ is convex with respect to the $\omega$ connexion. Then $g(t, (p, bY), s)$ is defined on $[-1, 1]$ and for $t = 1/2$ we claim $d(m, g(\alpha)) > s$ where $\alpha = (1/2, (p, bY), s)$, hence $g$ has reached a point at which $\bar{\omega} = \omega$ and hence may be indefinitely extended. The above inequality follows by letting $\beta = (1/2, (p, bY), 0)$, hence $\alpha$ and $\beta$ are both in $A$ and within $\delta_2$ of each other which implies $d(g(\alpha), g(\beta)) < R/16$. But $d(m, g(\beta)) \geq 3R/8$, hence

$$d(m, g(\alpha)) \geq \frac{3R}{8} - \frac{R}{16} = \frac{5R}{16} > \frac{R}{8} > s.$$ q.e.d.

We remark if we take $\bar{\omega} = \omega_1$ for any $0 < s' < s$ the conclusion of Theorem 1 still holds.

2. Attaching two spaces. Let $M$ and $M'$ be two $C^\infty$ manifolds, with the same dimension $n$, each carrying complete connexions. We wish to define a process which can roughly be described as follows. We remove two particular neighborhoods $V$ and $V'$ from $M$ and $M'$ respectively. Let $P = (M - V) \cup N \cup (M' - V')$ where $N$ is a "neck" attached to $M - V$ along the boundary of $V$ and similarly attached to $M' - V'$. We then define a complete connexion on $P$ which coincides with the original connexions in $M - V$ and $M' - V'$.

Using the function $\exp(-1/x^2)$ we may construct a regular univalent $C^\infty$ function $\phi : E^1 \to E^2$ such that for $t \geq 1$, $\phi(t) = (t, 0)$; for $t \leq -1$, $\phi(t) = (-t, 1)$; moreover, letting $\phi(t) = (x(t), y(t))$ we assume for $-1 < t < 1$ that $y(t)$ is strictly decreasing with $0 < y(t) < 1$; for $0 < t < 1$, $x(t)$ is strictly decreasing with $1/2 < x(t) < 1$, $x(0) = 1/2$; and $x(-t) = x(t)$ for $-1 < t < 0$.

For any real number $r > 0$ we define $N(r)$ to be the $n$-dimensional regular submanifold of $E^{n+1}$ consisting of the union of the three sets:

$$U_0 = \left[ (a_1, \cdots, a_n, 0) : \sum_{i=1}^{n} a_i^2 \geq r^2 \right],$$

$$U_1 = \left[ (a_1, \cdots, a_n, 1) : \sum_{i=1}^{n} a_i^2 \geq r^2 \right],$$

$$U_2 = \left[ (a_1x(t), \cdots, a_nx(t), y(t)) : \sum_{i=1}^{n} a_i^2 = r^2 \text{ and } -1 \leq t \leq 1 \right].$$
Using the Riemannian metric on $\mathcal{N}(r)$ induced from the Euclidean metric on $E^{n+1}$, we obtain a complete connexion $\omega_0$ on $\mathcal{N}(r)$ since $\mathcal{N}(r)$ is complete (closed) as a subset of $E^{n+1}$.

Let $M$ and $M'$ be $C^\infty$ manifolds of dimension $n$ carrying complete affine connexions $\omega$ and $\omega'$, respectively. Let $m, m'; U, U'; x_i, x_i'; R, R'$; and $d, d'$ all be chosen as in §1 for $M$ and $M'$ respectively. Applying Theorem 1 to $M$ and $M'$ we obtain a connexion $\tilde{\omega}$ on $M$ and number $s$, and a connexion $\tilde{\omega}'$ on $M'$ and number $s'$. By taking the smaller we may assume $s=s'$ by the remark at the end of §1.

Let $V=U(s/4), V'=U'(s/4)$. Let $N(s/4)$ be the submanifold of $E^{n+1}$ defined above. Let $N=\{p \text{ in } N(s/4): p \text{ in } U_2\}$. Let $P=(M-V)\cup N \cup (M'-V')$. If $p$ is in the boundary of $V$ we identify $p$ with the point in $N$ with coordinates $(x_1(p), \cdots, x_n(p), 0)$; similarly, for $p'$ in the boundary of $V'$ we identify $p'$ with the point in $N$ with coordinates $(x_1'(p), \cdots, x_n'(p'), 1)$. Our definition of $N$ assures us $P$ is a $C^\infty$ manifold. Let $\omega$ be the connexion defined on $P$ by $\omega=\tilde{\omega}$ on $M-V$, $\omega=\omega_0$ on $N$, $\omega=\tilde{\omega}'$ on $M'-V'$. By Theorem 1 these connexions agree on the "overlap." Furthermore $\omega$ is complete and $\omega=\omega_0$ on $M-U(s)$, $\omega=\omega'$ on $M'-U'(s)$.

We refer to the above process by saying we have attached $M$ and $M'$ through the neighborhoods $U$ and $U'$ respectively.

3. The example. Let $G$ be the Lie group $SL(2, C)=[2\times2 \text{ matrices } T \text{ over } C: \det T=1]$. Let $\omega$ be the complete connexion on $G$ defined by left translation. We provide two points $p_1$ and $p_2$ in $G$, with neighborhoods $U_1$ and $U_2$ respectively, such that any geodesic leaving $U_1$ must break at least once in order to intersect $U_2$. Then going to the simply connected covering and "inverting" through these neighborhoods we obtain the example.

We recall that the exponential map from the Lie algebra of $G$ into $G$ does not map onto $G$, for any element in the image has a square root and any element in this group having a square root has trace $\geq -2$, however there exist elements with trace $<-2$. Let $p$ be in $G$ with trace $p<-2$, let $e$ be the identity. Since trace is continuous we may let $U$ be a neighborhood of $p$ such that trace $(U)<-2$. We further take $U=V_1(p)$ where $V_1$ is a neighborhood of $e$. Let $V$ be a neighborhood of $e$ with $VV\subset V_1$ and $V=V^{-1}$. We claim no unbroken geodesic emanating from a point in $V$ intersects $Vp$. For if $g$ was such a geodesic, say $g(0)=m$ in $V$, $g(a)=p'$ in $Vp$, then $m^{-1}g(0)=e$, $m^{-1}g(a)=m^{-1}p'=m^{-1}m'p$ is in $U$. But $m^{-1}g$ is a geodesic and no unbroken geodesic passing through $e$ can intersect $U$ because of the trace condition. Let $U_1=V, U_2=Vp$.

For each integer $n\geq1$, let $M_n=G$. By §2 we may attach each $M_k$
to $M_{k+1}$, for $k \geq 1$, through the neighborhoods $U_2$ and $U_1$ respectively, obtaining a manifold $N$ with complete connexion which agrees with $\omega$ on each $M_k - (U_1 \cup U_2)$, $k > 1$. Hence for any integer $n$ we need only let $p_0 = e$ in $M_1 - U_2 \subset N$ and let $p_n$ be any point in $M_n - (U_1 \cup U_2)$, then any broken geodesic from $p_0$ to $p_n$ must break at least $n$ times. Finally, let $M$ be the simply connected covering of $N$, let $\pi: M \rightarrow N$ be the covering map, and define a connexion $\omega^*$ on $M$ by $\omega^* = \pi^* \omega$. For each $i \geq 0$, let $m_i$ be a point in $M$ with $\pi(m_i) = p_i$. Then any broken geodesic from $m_0$ to $m_n$ must contain at least $n$ breaks since $\pi$ carries a geodesic in $M$ into a geodesic in $G$.

**Bibliography**