THE NONTRIVIALITY OF THE RESTRICTION MAP IN THE COHOMOLOGY OF GROUPS

RICHARD G. SWAN

An unpublished result of B. Mazur states that if \( \pi \) is any non-trivial finite group then there is an \( i > 0 \) such that \( H^i(\pi, \mathbb{Z}) \neq 0 \). It is, of course, trivial that \( H^i(\pi, A) \neq 0 \) for some \( \pi \)-module \( A \). The point of Mazur's theorem is that we can even take \( A = \mathbb{Z} \), the ring of integers with trivial \( \pi \)-action. Mazur's proof of this theorem is geometric. It involves imbedding \( \pi \) in a compact Lie group \( G \) and studying the Leray-Cartan spectral sequence of the covering \( G \to G/\pi \).

The purpose of this paper is to prove the following theorem which generalizes Mazur's result.

**Theorem 1.** Let \( \pi \) be a finite group and \( \rho \) a nontrivial subgroup of \( \pi \). Then the restriction map \( i(\rho, \pi) : H^i(\pi, \mathbb{Z}) \to H^i(\rho, \mathbb{Z}) \) [2, Chapter XII, §8] is nonzero for an infinite number of values of \( i > 0 \).

As a consequence of this theorem, we get a generalization of Mazur's result.

**Corollary 1.** Let \( \pi \) be a finite group and let \( p \) be a prime dividing the order of \( \pi \). Then \( H^i(\pi, \mathbb{Z}) \) has a nonzero \( p \)-primary component for an infinite number of values of \( i > 0 \).

To see this we have merely to use Theorem 1, choosing for \( \rho \) any nontrivial \( p \)-group in \( \pi \).

The proof of Theorem 1 will also be geometric. In fact, I will actually prove the following much more general theorem whose proof must necessarily be geometric.

**Theorem 2.** Let \( G \) be a compact, not necessarily connected Lie group. Let \( H \) be a closed nontrivial subgroup of \( G \), also not necessarily connected. Let \( f : BH \to BG \) be the map of classifying spaces induced by the inclusion map \( H \to G \) [1, §1]. Then \( f^* : H^i(BG, \mathbb{Z}) \to H^i(BH, \mathbb{Z}) \) is nonzero for an infinite number of values of \( i \).

**Remark.** If \( H \) has an element of order \( p \), the proof of this theorem will also show that \( f^* : H^i(BG, \mathbb{Z}_p) \to H^i(BH, \mathbb{Z}_p) \) is nontrivial for an infinite number of values of \( i \). If \( H \) is infinite, it will show that
$f^*: H^i(B_G, Q) \to H^i(B_H, Q)$ is nonzero for an infinite number of values of $i$. Here $Q$ is the field of rational numbers.

**Proof.** By the Peter-Weyl theorem $G$ has a faithful unitary representation [4, Chapter VI, Theorem 4] and so can be imbedded in a unitary group $U(n)$. Also, $H$ has a subgroup isomorphic to $Z_p$ for some prime $p$. This is trivial if $H$ is finite, but if $H$ is infinite it contains a torus [3, Exposé 23, Theorem 1] which clearly has a subgroup isomorphic to $Z_p$. Since the map $B_{Z_p} \to B_{U(n)}$ factors through $f$, it will be sufficient to prove the theorem for the case $G \cong Z_p$ and $G \cong U(l)$. (If $H$ is infinite and we are trying to show that $H^i(B_G, Q) \to H^i(B_H, Q)$ is nontrivial, it will suffice to consider the case where $G \cong U(n)$ and $H$ is a circle group. The rest of the proof will be substantially the same.)

Assume then that $H \cong Z_p$, $G \cong U(l)$. Imbed $H$ in a maximal torus $T$ of $G$. This can be done by taking any maximal torus $T$ containing a generator of $H$ [3, Exposé 23, Theorem 1]. Now, $H^*(B_T, Z)$ is a polynomial ring over $Z$ with generators $t_1, \ldots, t_l \in H^2(B_T, Z)$. The image of $H^*(B_G, Z)$ in $H^2(B_T, Z)$ consists of all symmetric polynomials in $t_1, \ldots, t_l$ [1, §4]. Therefore to prove the theorem it will be sufficient to find sufficiently many symmetric polynomials which map nontrivially into $H^*(B_H, Z)$ under the map $g^*$ induced by $g: B_H \to B_T$. This map $g$ is, of course, induced by the inclusion $H \to T$.

Now, $H^*(B_H, Z)$ is a polynomial ring over $Z_p$ with a single generator $\alpha \in H^2(B_H, Z)$ [2, Chapter XII, §7]. Therefore $g^*(t_i) = r_i \alpha$ with $r_i \in Z_p$. I claim that at least one $r_i \neq 0$. Suppose to the contrary that all $r_i = 0$. Then $g^*: H^2(B_T, Z) \to H^2(B_H, Z)$ must be zero. Now $g: B_H \to B_T$ is a fiber map with fiber $T/H$ [1, §1]. Of course, $T/H$ is a torus, being a connected abelian Lie group. The map $g^*: H^2(B_T, Z) \to H^2(B_H, Z)$ is just the map $E_2^0 \to E_2^0$ in the spectral sequence of this fibration. If it is zero, all elements of $E_2^0$ must be trivial. Therefore $d_2: E_2^{0,1} \to E_2^{2,0}$ must be onto. This shows that $T/H$ has rank $l$ and that $H_1(T/H, Z) = E_2^{0,1}$ has a base $\{x_\nu\}$ such that $d_2 x_\nu = t_\nu$. (Of course it is trivial that $T/H$ has rank $l$, $H$ being finite, but I have arranged the proof so that it works for $H = S^1$ without essential change.) Now $E_2^{0,2} = H^2(T/H, Z)$ has a base $x_\mu x_\nu$, with $\mu < \nu$. Since $d_2$ is a derivation, $d_2(x_\mu x_\nu) = t_\mu \otimes x_\nu - t_\nu \otimes x_\mu$ in $E_2^{2,1} = H^2(B_T) \otimes H^1(T/H)$. Since these elements are linearly independent in $E_2^{2,1}$, $d_2$ is a monomorphism on $E_2^{0,2}$ and so $E_2^{0,2} = 0$. Also $E_2^{2,0} = 0$ and $E_2^{1,1} = 0$. Thus the spectral sequence shows that $H^2(B_H, Z) = 0$ which is absurd.

Now let $s$ be the number of indices $\nu$ for which $r_\nu \neq 0$. By renumbering we can assume that $r_\nu \neq 0$ for $\nu = 1, 2, \ldots, s$ and $r_\nu = 0$ for $\nu > s$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Let $x$ be the $s$th elementary symmetric function in $t_1, \ldots, t_l$. Then, for $k > 0$,

$$g^*(x^k) = \left( \prod_{1}^{s} r_r \right)^k \alpha x^k \neq 0.$$ 

Since the $x^k$ are symmetric polynomials and have arbitrarily large dimensions, this proves the theorem.

**Remark.** If $l$ is the smallest dimension of a faithful representation of $G$ over the complex numbers, the proof shows that $f^*: H^i(B_\alpha, Z) \to H^i(B_H, Z)$ is nonzero for some $i \leq 2l$ (since $i = 2s$ and $s \leq l$). This is a best possible result if no further conditions are placed on $G$, $H$ and $l$. To see this for finite groups, let $H$ be the cyclic group of order $p$ permuting $p$ symbols and let $G$ be the normalizer of $H$ in the symmetric group $S_p$.

If $R$ denotes the real numbers, duality shows that $f^*: H_i(B_H, R/Z) \to H_i(B_H, R/Z)$ is nonzero for an infinite number of values of $i$. But, if $\pi$ is finite, $H_i(\pi, R/Z) \approx H_{i-1}(\pi, Z)$, cf. [2, Chapter XII, Proof of Theorem 6.6]. Therefore Theorem 1 has the following corollary.

**Corollary 2.** Let $\pi$ be a finite group and $\rho$ a nontrivial subgroup of $\pi$. Then the induced map $H_i(\rho, Z) \to H_i(\pi, Z)$ is nontrivial for an infinite number of values of $i > 0$.

Equivalently, we may say that the transfer $i(\pi, \rho): H^i(\rho, Z) \to H^i(\pi, Z)$ is nonzero for an infinite number of negative values of $i$ [2, Chapter XII, Exercise 8].

Note that the example $Z_p \subset Z_p + Z_p$ shows that the restriction map can be zero in all negative dimensions and the transfer zero in all positive dimensions.

It would be interesting to have a purely algebraic proof of Theorem 1 but I know of no such proof.

**References**