A COUNTEREXAMPLE OF KOEBE'S FOR SLIT MAPPINGS

EDGAR REICH

1. We refer to a region $\Omega$ of the extended $z$-plane as a (parallel) slit domain if $\infty \in \Omega$, and if the components of the boundary, $\partial \Omega$, are either points, or segments ("slits") parallel to a common line, which without loss of generality will be assumed to be the $y$-axis ($z=x+iy$). It was originally conjectured by Koebe that if two slit domains $\Omega_1$ and $\Omega_2$ are conformally equivalent, that is, if there exists a function $f$, schlicht in $\Omega_2$, such that $f(\infty) = \infty$, $f(\Omega_2) = \Omega_1$, then, unless $f$ is linear, at most one of the two sets, $E_1 = \partial \Omega_1$, $E_2 = \partial \Omega_2$, has area zero. Later on Koebe [5] outlined the construction of a counterexample in which (using the present notation)

(a) the components of $E_1$ are not all points,
(b) the projection of $E_1$ onto the $x$-axis has linear Lebesgue measure zero,
(c) $E_2$ is a compact, totally disconnected subset of the $x$-axis.

Although Koebe's example, and variants therefore, have been applied repeatedly in connection with various counterexamples in complex variable theory (see, for instance, [7]) it does not appear to have been previously noted in the literature that the reasoning in [5] contains a gap. The statement containing the word "offenbar" in the last paragraph of page 62 of [5] is incorrect. If $P$ denotes the intersection of Koebe's $\Omega_1$ with a line parallel to the $y$-axis, then $P$ is denumerable, and supposedly closed. However, it is not difficult to show that the set of points in $P$ that are two-sided limit points of $P$ must be dense in itself. In the present note we fill this gap by obtaining the following slightly more general result.

**Theorem 1.** Let $A$ be a bounded, perfect, nowhere dense, linear set. There exist conformally equivalent slit domains $\Omega_1$, $\Omega_2$ whose boundaries $E_i = \partial \Omega_i$, $i = 1, 2$, have the following properties:

(i) The components of $E_1$ are not all points.
(ii) The projection of $E_1$ onto the $x$-axis is $A$.
(iii) $E_2$ is a compact, totally disconnected subset of the $x$-axis.

To obtain an example for which (b) holds one then merely chooses

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A to have linear measure zero, for example, the Cantor middle-third set.

To construct $E_1$ and $E_2$ we shall first obtain some auxiliary results in §§2 and 3. The sets $E_1$ and $E_2$ are described in §4. Our construction follows the ideas of Koebe, the principal deviation from Koebe’s work being the method of construction of the set here denoted by $\Sigma$.

Before proceeding with the proof of Theorem 1, it is of interest to note an immediate corollary. A significant property of the set $E_2$ of Theorem 1 can be stated in modern terminology if, following [1], we say that a compact set $E$ in the $z$-plane is a null-set of class $\mathcal{N}_D$ if and only if, for any region $Z$ of the $z$-plane containing $E$, every function $f(z)$, regular in $Z - E$, and possessing a finite Dirichlet integral there, can be extended to a function regular in $Z$. (The class $\mathcal{N}_D$ has been studied extensively by Sario [7], Ahlfors and Beurling [1], and others.) It is known that if $E$ is a null-set of class $\mathcal{N}_D$ then the boundary of any region conformally equivalent to the complement of $E$ with respect to the extended $z$-plane must be totally disconnected. Now $E_1$ contains a continuum, by (i). Hence, using the example $E = E_2$, with $A$ arbitrary, we have the following.

**Theorem 2.** There exists a linear, compact, totally disconnected set $E$ which is not a null-set of class $\mathcal{N}_D$.

2. In what follows $R$ denotes the real line. Given a set $A \subset R$, let $A_\infty = \{ x \mid x \in A, x$ is the limit of points of $A$ both from the left and right $\}$. Let $A_I = A - A_\infty$.

**Lemma 1.** If $A$ is a perfect set in $R$, then $A_I$ contains a perfect set.

**First Proof.** Let $\{ I_i \}, i = 1, 2, \cdots$ be a (possibly terminating) enumeration of the components of $R - A$. Each set $I_i$ is a finite or semi-infinite open interval. Note that $x \in A_I$ if and only if $x$ is a one-sided limit point of $A$, that is, if and only if $x$ is the finite endpoint of some interval $I_i$. Hence $A_I$ is a finite or denumerable set. Since $A$ is uncountable, $A_\infty = A - A_I$ is an uncountable Borel set. Therefore [2] $A_\infty$ contains a perfect set.

**Second Proof.** Each finite endpoint of $I_i$ is in $A_I$. On the other hand, each point of $A$ is a condensation point. Therefore each finite endpoint of $I_i$ is the limit of points of $A_\infty \cap (R - I_i)$. We shall use this remark in the course of defining a (possibly terminating) sequence $\{ J_i \}, i = 1, 2, \cdots$, of finite or semi-infinite open intervals, below. The definition is by induction.

(a) $J_1$ is an open interval whose finite endpoints are in $A_\infty$, and such that $J_1 \supset \text{Cl}[I_1]$. (The symbol Cl denotes closure.)
(b) Suppose \( J_1, \cdots, J_n \) have been determined with the following properties:

(b1) \( \text{Cl}[J_i] \cap \text{Cl}[J_j] = \emptyset \), unless \( i=j \), \( i,j=1,2,\cdots,n \),

(b2) Each finite endpoint of \( J_k, k=1,2,\cdots,n \), is in \( A_{II} \). (Consequently, by (b1), each \( \text{Cl}[I_i] \) is either in some \( J_k, 1 \leq k \leq n \), or in \( R - U^n J_i \).

If all \( \text{Cl}[I_i] \) are in \( U^n J_i \) the process terminates. Otherwise, let \( I_{i(n)} \) be the first interval in the enumeration \( \{ I_i \} \) whose closure lies in \( R - U^n J_i \). In view of the remark at the beginning of the proof we can choose \( J_{n+1} \), disjoint from \( U^n \text{Cl}[J_i] \), in such a way that each finite endpoint of \( J_{n+1} \) is in \( A_{II} \), and \( J_{n+1} \subseteq \text{Cl}[I_{i(n)}] \).

The sequence \( \{ J_i \} \) resulting from the above process has the following properties.

(i) Each \( J_i \) is an open, finite or semi-infinite interval in \( R \).

(ii) \( \text{Cl}[J_i] \cap \text{Cl}[J_j] = \emptyset \), unless \( i=j \).

(iii) Each set \( \text{Cl}[I_i] \) is contained in some \( J_j \).

Conclusion (iii) is assured because the sequence \( i(1), i(2), \cdots \) is strictly increasing. By (i) and (ii), \( R - U J_i \) is a nonempty closed set. By (ii), this set has no isolated points, hence is perfect. By (iii), \( R - U J_i \subseteq R - \text{Cl}[J_i] = A_{II} \), as required. For future reference we note that the sequence \( \{ J_i \} \) can be constructed in such a manner that the number of semi-infinite intervals among \( \{ J_i \} \) that are semi-infinite to the left (right) does not exceed the number of semi-infinite intervals among \( \{ I_i \} \) that are semi-infinite to the left (right).

**Lemma 2.** If \( A \) and \( B \) are perfect sets in \( R \), and \( A_{II} \supseteq B \), then there exists a perfect set \( C \) such that \( A_{II} \supseteq C \supseteq C_{II} \supseteq B \).

**Proof.** Let \( K \) be a component of \( R - B \), and let \( \{ I_i(K) \} \) be an enumeration of the components of \( K - A \). (\( K - A \) may be empty, in which case the enumeration is also empty.) \( A \cap K \) is a perfect set in the topology relative to \( K \). Since \( K \) is homeomorphic with \( R \), it follows from statements (i) to (iv) of the proof of Lemma 1 that there is a sequence \( \{ J_i(K) \} \) of subintervals of \( K \) such that, in the topology relative to \( K \),

(i') each \( J_i(K) \) is an open subinterval of \( K \) whose finite endpoints fall in \( K \),

(ii') \( \text{Cl}[J_i(K)] \cap \text{Cl}[J_j(K)] = \emptyset \), unless \( i=j \),

(iii') each \( \text{Cl}[I_i(K)] \) is contained in some \( J_j(K) \).

Conclusion (i') follows from (i) and (iv) of Lemma 1, because the finite endpoints of \( K \) lie in \( B \), and hence also in \( A_{II} \). This implies that no endpoint of any \( I_i(K) \) coincides with a finite endpoint of \( K \). Therefore, by (iv), translated to the present case by means of the homeomorphism, it follows that all \( J_i(K) \) can be chosen so that (i')
holds. In view of (i'), it actually follows that (i') to (iii') hold in the topology of \( R \).

If \( \{ K_i \} \) is an enumeration of the components of \( R - B \) then the set \( C = R - \bigcup_{i,j} J_i(K_j) \) is perfect, by (i') and (ii'). Moreover, \( C_{II} = R - \bigcup_{i,j} \text{Cl}[J_i(K_j)] \supset B \), since \( \text{Cl}[J_i(K_j)] \subset K_j \), by (i'). Also, \( C \subset C_{II} \), by (iii'), as required.

**Lemma 3.** Suppose \( A \) is a perfect set in \( R \), and \( H \) is a countable set, \( H \subset [0, 1] \), \( 0 \in H \), \( 1 \in H \). There exists a sequence of perfect sets \( \{ A(y) \} \), \( y \in H \), such that

\[
A(0) = A, \quad A(z) \subset A(y)_{II} \text{ whenever } y < z, y \in H, z \in H.
\]

If \( A \) is totally disconnected, then \( A(y) \) is totally disconnected for each \( y \in H \).

**Proof.** Suppose \( H = \{ y_i \} \), \( i = 1, 2, \cdots, 0 = y_1, 1 = y_2 \). To construct \( \{ A(y) \} \) for \( y \in H \) we start with \( A(0) = A \). Lemma 1 guarantees the existence of a set \( A(1) \subset A(0)_{II} \). Let \( z_i, w_i \) be the unique numbers among \( y_1, y_2, \cdots, y_i \) closest to \( y_{i+1} \), with \( z_i < y_{i+1} < w_i \). To construct \( A(y_{i+1}) \), \( i \geq 2 \), we apply Lemma 2, with \( A(y_{i+1}) \) corresponding to \( C \), interpreting the \( A \) of Lemma 2 as \( A(z_i) \), and the \( B \) of Lemma 2 as \( A(w_i) \). Since \( y \in H \) implies \( A(y) \subset A \), it is clear that \( A(y) \) is totally disconnected for each \( y \in H \), if \( A \) is totally disconnected.

3. Let \( A \) be any perfect, totally disconnected set in \([0, 1]\). Using Lemma 3, we shall construct a closed set \( \Sigma \) lying in the unit square \( \{ x + iy \mid 0 \leq x \leq 1, 0 \leq y \leq 1 \} \) of the \( z \)-plane \( (z = x + iy) \) with the following properties.

(a) The components of \( \Sigma \) are closed linear segments (possibly points) \( L_\xi \), parallel to the \( y \)-axis, with lower initial point on the \( x \)-axis, at \( x = \xi, \xi \in A \).

(b) There is at least one component \( L_\xi \) (actually uncountably many) with length \( |L_\xi| > 0 \).

(c) Any point in \( \Sigma \), not at the top of a segment \( L_\xi \), is the limit, both from the left and right, of points of \( \Sigma \).

To construct \( \Sigma \) we identify the \( A \) of Lemma 3 with our present \( A \), and choose as the \( H \) of Lemma 3 a countable set, such as the rationals in \([0, 1]\), whose closure is \([0, 1]\). The sets \( A(y) \) of Lemma 3 will be employed essentially as cross-sections of our set \( \Sigma \). Namely, we define

\[
\Sigma = \bigcup_{\xi \in A} L_\xi,
\]

where \( L_\xi \) is a closed vertical segment (possibly a point) with lower initial point on the \( x \)-axis, at \( x = \xi \), and
\[ |L_z| = \sup\{y \mid x \in A(y), y \in H\}. \]

To show that \( \Sigma \) is closed, suppose \( z_0 = x_0 + iy_0 \in \text{Cl}\{\Sigma}\). Then there exist points \( x_n + iy_n \in \Sigma, n = 1, 2, \ldots \), such that \( x_0 = \lim x_n, y_0 = \lim y_n \). Since \( x_n + iy_n \in \Sigma \) implies \( x_n \in A \), it follows that \( x_0 \in A \), because \( A \) is closed. The case \( y_0 = 0 \) is immediately disposed of by noting that, since \( x_0 \in A, z_0 = x_0 + i0 \in \Sigma \). If \( y_0 > 0 \), let \( \delta > 0 \) be arbitrary. By the density property of \( H \), there exist \( \delta', \delta'' \), \( 0 < \delta' < \delta'' < \delta \), such that \( y_0 - \delta'' \in H \). Now \( y_0 > y_0 - \delta' \) for \( n > N(\delta') \). Therefore, from the definition of \( \Sigma \), \( x_n + i(y_0 - \delta') \in \Sigma \) for \( n > N(\delta') \). Therefore, \( x_n \in A(y_0 - \delta'') \) for \( n > N(\delta') \). Since \( A(y_0 - \delta'') \) is closed, \( x_0 \in A(y_0 - \delta'') \). Thus \( |L_{z_0}| \geq |y_0 - \delta''| - y_0 - \delta \). Hence \( |L_{z_0}| \geq y_0 \). Therefore, \( x_0 + iy_0 \in \Sigma \).

A point is in \( \Sigma \) if and only if it lies on some segment \( L_\xi \). Hence, if a component \( K \) of \( \Sigma \) contained \( L_\xi \cup L_\eta \), \( \xi < \eta \), then \( K \) would have to contain \( \{x + iy \mid \xi \leq x \leq \eta, y = 0\} \). But this is impossible, because \( A \) is totally disconnected. Hence we have property (a).

Property (b) follows from the fact that \( A(1) \) is perfect, hence contains at least one point (actually uncountably many), say \( \lambda \in A(1) \). Thus, \( |L_\lambda| = 1, L_\lambda \subset \Sigma \).

To prove (c), assume \( x_0 + iy_0 \in \Sigma, y_0 < |L_{z_0}| \). Let us choose \( \eta_1, \eta_2 \in H, y_0 < \eta_1 < \eta_2 < |L_{z_0}| \). We have \( x_0 \in A(\eta_2) \). Therefore, \( x_0 \in A(\eta_2) \). Therefore, there exist sequences \( \{\xi_n\}, \{x_n\} \), of points of \( A(\eta_1) \) such that \( \xi_n < x_n < x_n \), \( \lim \xi_n = \lim x_n = x_0 \). Property (c) follows from the fact that \( x_n + iy_0 \) and \( x_n + iy_0 \) lie in \( \Sigma \).

4. Let \( \Omega \) be the region
\[ \{z \mid \Im z > 0\} - \Sigma, \quad (z = x + iy), \]
where \( \Sigma \) is the set found in §3. By the Riemann mapping theorem there exists a schlicht function, \( w = f(z) \), \( f(\infty) = \infty \), mapping \( \Omega \) onto the upper half \( w \)-plane. Since the limiting values of \( f \) on the linear open set \( R - A \) of the \( x \)-axis are real, it follows, by the strong form of the Schwarz reflection principle, that there is an extension of \( f \) to a function schlicht in
\[ \Omega_1 = \Omega \cup \Omega^* \cup \{z \mid \Im z = 0, \Re z \in R - A\} \cup \{\infty\}, \]
with \( f(\infty) = \infty \), \( f(z^*) = f(z)^* \). (* denotes reflection in the real axis.) Evidently, \( E_1 = \partial \Omega_1 = \Sigma \cup \Sigma^* \). The set of limiting values, \( E_2 \), of \( f \) on \( E_1 \) is real, and according to Carathéodory's prime end theory [3], if \( \Omega_0 = f(\Omega_1) \), then \( E_2 = \partial \Omega_0 \). In view of the reflection principle, \( E_2 \) is also the set of limiting values on \( \Sigma \) of the restriction of \( f \) to \( \Omega \). By property (c), §3, each segment \( L_x, x \in A \), with \( |L_x| > 0 \), is the impression of a prime end (of the second type) of \( \partial \Omega \). The remaining points of \( \Sigma \) (all
on the x-axis) are accessible points of \( \partial \Omega \). Furthermore, since \( A \) is totally disconnected, there are accessible points of \( \partial \Omega \) in \( R - A \) between any two prime ends of \( \Sigma \). Hence, by Carathéodory's theory, \( E_2 \) is a totally disconnected set. This completes the proof of Theorem 1.

The fact that \( E_2 \) is totally disconnected makes it possible to interpret \( E_2 \) not only as the boundary of a (parallel) slit domain, but also, for instance, as the boundary of a circular slit domain. For example, by deleting \( E_2 \) along a radius from an annulus (or disk) one obtains an example of a circular slit annulus (or disk) which can be shown to be not minimal in the sense of [4] or [6].

References


University of Minnesota