A COUNTEREXAMPLE OF KOEBE'S FOR SLIT MAPPINGS¹

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1. We refer to a region $\Omega$ of the extended $z$-plane as a (parallel) slit domain if $\infty \in \Omega$, and if the components of the boundary, $\partial \Omega$, are either points, or segments ("slits") parallel to a common line, which without loss of generality will be assumed to be the $y$-axis ($z = x + iy$). It was originally conjectured by Koebe that if two slit domains $\Omega_i$ and $\Omega_2$ are conformally equivalent, that is, if there exists a function $f$, schlicht in $\Omega_2$, such that $f(\infty) = \infty$, $f(\Omega_2) = \Omega_1$, then, unless $f$ is linear, at most one of the two sets, $E_1 = \partial \Omega_i$, $E_2 = \partial \Omega_2$, has area zero. Later on Koebe [5] outlined the construction of a counterexample in which (using the present notation)

(a) the components of $E_1$ are not all points,
(b) the projection of $E_1$ onto the $x$-axis has linear Lebesgue measure zero,
(c) $E_2$ is a compact, totally disconnected subset of the $x$-axis.

Although Koebe's example, and variants therefore, have been applied repeatedly in connection with various counterexamples in complex variable theory (see, for instance, [7]) it does not appear to have been previously noted in the literature that the reasoning in [5] contains a gap. The statement containing the word "offenbar" in the last paragraph of page 62 of [5] is incorrect. If $P$ denotes the intersection of Koebe's $\Omega_i$ with a line parallel to the $y$-axis, then $P$ is denumerable, and supposedly closed. However, it is not difficult to show that the set of points in $P$ that are two-sided limit points of $P$ must be dense in itself. In the present note we fill this gap by obtaining the following slightly more general result.

Theorem 1. Let $A$ be a bounded, perfect, nowhere dense, linear set. There exist conformally equivalent slit domains $\Omega_1$, $\Omega_2$ whose boundaries $E_i = \partial \Omega_i$, $i = 1$, 2, have the following properties:

(i) The components of $E_1$ are not all points.
(ii) The projection of $E_1$ onto the $x$-axis is $A$.
(iii) $E_2$ is a compact, totally disconnected subset of the $x$-axis.

To obtain an example for which (b) holds one then merely chooses

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A to have linear measure zero, for example, the Cantor middle-third set.

To construct $E_1$ and $E_2$ we shall first obtain some auxiliary results in §§2 and 3. The sets $E_1$ and $E_2$ are described in §4. Our construction follows the ideas of Koebe, the principal deviation from Koebe's work being the method of construction of the set here denoted by $\Sigma$.

Before proceeding with the proof of Theorem 1, it is of interest to note an immediate corollary. A significant property of the set $E_2$ of Theorem 1 can be stated in modern terminology if, following [1], we say that a compact set $E$ in the $z$-plane is a null-set of class $N_D$ if and only if, for any region $Z$ of the $z$-plane containing $E$, every function $f(z)$, regular in $Z - E$, and possessing a finite Dirichlet integral there, can be extended to a function regular in $Z$. (The class $N_D$ has been studied extensively by Sario [7], Ahlfors and Beurling [1], and others.) It is known that if $E$ is a null-set of class $N_D$ then the boundary of any region conformally equivalent to the complement of $E$ with respect to the extended $z$-plane must be totally disconnected. Now $E_1$ contains a continuum, by (i). Hence, using the example $E = E_2$, with $A$ arbitrary, we have the following.

**Theorem 2.** There exists a linear, compact, totally disconnected set $E$ which is not a null-set of class $N_D$.

2. In what follows $R$ denotes the real line. Given a set $A \subset R$, let $A_n = \{x : x \in A, x$ is the limit of points of $A$ both from the left and right}. Let $A_I = A - A_n$.

**Lemma 1.** If $A$ is a perfect set in $R$, then $A_I$ contains a perfect set.

**First Proof.** Let $\{I_i\}, i = 1, 2, \cdots$ be a (possibly terminating) enumeration of the components of $R - A$. Each set $I_i$ is a finite or semi-infinite open interval. Note that $x \in A_I$ if and only if $x$ is a one-sided limit point of $A$, that is, if and only if $x$ is the finite endpoint of some interval $I_i$. Hence $A_I$ is a finite or denumerable set. Since $A$ is uncountable, $A_I = A - A_I$ is an uncountable Borel set. Therefore $[2] A_I$ contains a perfect set.

**Second Proof.** Each finite endpoint of $I_i$ is in $A_I$. On the other hand, each point of $A$ is a condensation point. Therefore each finite endpoint of $I_i$ is the limit of points of $A_I \cap (R - I_i)$. We shall use this remark in the course of defining a (possibly terminating) sequence $\{J_i\}, i = 1, 2, \cdots$, of finite or semi-infinite open intervals, below. The definition is by induction.

(a) $J_1$ is an open interval whose finite endpoints are in $A_I$, and such that $J_1 \supset Cl[I_1]$. (The symbol $Cl$ denotes closure.)
(b) Suppose \( J_1, \ldots, J_n \) have been determined with the following properties:

(b1) \( \text{Cl}[J_i] \cap \text{Cl}[J_j] = \emptyset \), unless \( i=j \), \( i, j = 1, 2, \ldots, n \),

(b2) Each finite endpoint of \( J_k, k = 1, 2, \ldots, n \), is in \( A_{II} \). (Consequently, by (b1), each \( \text{Cl}[I_i] \) is either in some \( J_k, 1 \leq k \leq n \), or in \( R-U_{J_i} \).

If all \( \text{Cl}[I_i] \) are in \( U_{J_i} \) the process terminates. Otherwise, let \( I_{i(n)} \) be the first interval in the enumeration \( \{ I_i \} \) whose closure lies in \( R-U_{J_i} \). In view of the remark at the beginning of the proof we can choose \( J_{n+1} \), disjoint from \( U_{I_i} \text{Cl}[J_i] \), in such a way that each finite endpoint of \( J_{n+1} \) is in \( A_{II} \), and \( J_{n+1} \supset \text{Cl}[I_{i(n)}] \).

The sequence \( \{ J_i \} \) resulting from the above process has the following properties.

(i) Each \( J_i \) is an open, finite or semi-infinite interval in \( R \).

(ii) \( \text{Cl}[J_i] \cap \text{Cl}[J_j] = \emptyset \), unless \( i=j \).

(iii) Each set \( \text{Cl}[I_i] \) is contained in some \( J_j \).

Conclusion (iii) is assured because the sequence \( i(1), i(2), \ldots \) is strictly increasing. By (i) and (ii), \( R-U_{J_i} \) is a nonempty closed set. By (ii), this set has no isolated points, hence is perfect. By (iii), \( R-U_{J_i} \subset R-\text{Cl}[J_i] = A_{II} \), as required. For future reference we note that the sequence \( \{ J_i \} \) can be constructed in such a manner that

(iv) the number of semi-infinite intervals among \( \{ J_i \} \) that are semi-infinite to the left (right) does not exceed the number of semi-infinite intervals among \( \{ I_i \} \) that are semi-infinite to the left (right).

Lemma 2. If \( A \) and \( B \) are perfect sets in \( R \), and \( A_{II} \supset B \), then there exists a perfect set \( C \) such that \( A_{II} \supset C \supset C_{II} \supset B \).

Proof. Let \( K \) be a component of \( R-B \), and let \( \{ I_i(K) \} \) be an enumeration of the components of \( K-A \). (\( K-A \) may be empty, in which case the enumeration is also empty.) \( A \cap K \) is a perfect set in the topology relative to \( K \). Since \( K \) is homeomorphic with \( R \), it follows from statements (i) to (iv) of the proof of Lemma 1 that there is a sequence \( \{ J_i(K) \} \) of subintervals of \( K \) such that, in the topology relative to \( K \),

(i') each \( J_i(K) \) is an open subinterval of \( K \) whose finite endpoints fall in \( K \),

(ii') \( \text{Cl}[J_i(K)] \cap \text{Cl}[J_j(K)] = \emptyset \), unless \( i=j \),

(iii') each \( \text{Cl}[I_i(K)] \) is contained in some \( J_i(K) \).

Conclusion (i') follows from (i) and (iv) of Lemma 1, because the finite endpoints of \( K \) lie in \( B \), and hence also in \( A_{II} \). This implies that no endpoint of any \( I_i(K) \) coincides with a finite endpoint of \( K \). Therefore, by (iv), translated to the present case by means of the homeomorphism, it follows that all \( J_i(K) \) can be chosen so that (i')
holds. In view of (i'), it actually follows that (i') to (iii') hold in the topology of \( R \).

If \( \{ K_i \} \) is an enumeration of the components of \( R - B \) then the set \( C = R - \bigcup_{i,j} J_i(K_j) \) is perfect, by (i') and (ii'). Moreover, \( C_{II} = R - \bigcup_{i,j} \text{Cl}[J_i(K_j)] \supseteq B \), since \( \text{Cl}[J_i(K_j)] \subseteq K_j \), by (i'). Also, \( C \subseteq A_{II} \), by (iii'), as required.

**Lemma 3.** Suppose \( A \) is a perfect set in \( R \), and \( H \) is a countable set, \( H \subseteq [0, 1] \), \( 0 \in H \), \( 1 \in H \). There exists a sequence of perfect sets \( \{ A(y) \} \), \( y \in H \), such that

\[
A(0) = A, \quad A(z) \subseteq A(y)_{II} \text{ whenever } y < z, \ y \in H, \ z \in H.
\]

If \( A \) is totally disconnected, then \( A(y) \) is totally disconnected for each \( y \in H \).

**Proof.** Suppose \( H = \{ y_i \}, i = 1, 2, \ldots, 0 = y_1, 1 = y_2 \). To construct \( \{ A(y) \} \) for \( y \in H \) we start with \( A(0) = A \). Lemma 1 guarantees the existence of a set \( A(1) \subseteq A(0)_{II} \). Let \( z_i \), \( w_i \) be the unique numbers among \( y_1, y_2, \ldots, y_i \) closest to \( y_{i+1} \), with \( z_i < y_{i+1} < w_i \). To construct \( A(y_{i+1}) \), \( i \geq 2 \), we apply Lemma 2, with \( A(y_{i+1}) \) corresponding to \( C \), interpreting the \( A \) of Lemma 2 as \( A(z_i) \), and the \( B \) of Lemma 2 as \( A(w_i) \). Since \( y \in H \) implies \( A(y) \subseteq A \), it is clear that \( A(y) \) is totally disconnected for each \( y \in H \), if \( A \) is totally disconnected.

3. Let \( A \) be any perfect, totally disconnected set in \([0, 1]\). Using Lemma 3, we shall construct a closed set \( \Sigma \) lying in the unit square \( \{ x + iy | 0 \leq x \leq 1, 0 \leq y \leq 1 \} \) of the \( z \)-plane \( (z = x + iy) \) with the following properties.

(a) The components of \( \Sigma \) are closed linear segments (possibly points) \( L_\xi \), parallel to the \( y \)-axis, with lower initial point on the \( x \)-axis, at \( x = \xi, \ \xi \in A \).

(b) There is at least one component \( L_\xi \) (actually uncountably many) with length \( |L_\xi| > 0 \).

(c) Any point in \( \Sigma \), not at the top of a segment \( L_\xi \), is the limit, both from the left and right, of points of \( \Sigma \).

To construct \( \Sigma \) we identify the \( A \) of Lemma 3 with our present \( A \), and choose as the \( H \) of Lemma 3 a countable set, such as the rationals in \([0, 1]\), whose closure is \([0, 1]\). The sets \( A(y) \) of Lemma 3 will be employed essentially as cross-sections of our set \( \Sigma \). Namely, we define

\[
\Sigma = \bigcup_{\xi \in A} L_\xi,
\]

where \( L_\xi \) is a closed vertical segment (possibly a point) with lower initial point on the \( x \)-axis, at \( x = \xi \), and
To show that $\Sigma$ is closed, suppose $z_0 = x_0 + iy_0 \in \Omega \Sigma \Sigma$. There then exist points $x_n + iy_n \in \Sigma, n = 1, 2, \ldots$, such that $x_0 = \lim x_n, y_0 = \lim y_n$. Since $x_n + iy_n \in \Sigma$, it follows that $x_0 \in A$, because $A$ is closed. The case $y_0 = 1$ is immediately disposed of by noting that, since $x_0 \in A, z_0 = x_0 + i0 \in \Sigma$. If $y_0 > 1$, let $\delta > 1$ be arbitrary. By the density property of $H$, there exist $\delta', \delta''$, $0 < \delta' < \delta'' < \delta$, such that $y_0 - \delta'' \in H$. Now $y_n > y_0 - \delta'$ for $n > N(\delta')$. Therefore, from the definition of $\Sigma$, $x_n + i(y_0 - \delta') \in \Sigma$ for $n > N(\delta')$. Since $A(y_0 - \delta'')$ is closed, $x_0 \in A(y_0 - \delta'')$. Thus $|L_{z_0}| \geq y_0 - \delta'' > y_0 - \delta$. Hence $L_{z_0} \geq y_0$. Therefore, $x_0 + iy_0 \in \Sigma$.

A point is in $\Sigma$ if and only if it lies on some segment $L_\xi$. Hence, if a component $K$ of $\Sigma$ contained $L_\xi \cup L_{\eta}, \xi < \eta$, then $K$ would have to contain \{x + iy \mid \xi < x < \eta, y = 0\}. But this is impossible, because $A$ is totally disconnected. Hence we have property (a).

Property (b) follows from the fact that $A(1)$ is perfect, hence contains at least one point (actually uncountably many), say $\lambda \in A(1)$. Thus, $|L_\lambda| = 1, L_\lambda \subset \Sigma$.

To prove (c), assume $z_0 + iy_0 \in \Sigma, y_0 < |L_{z_0}|$. Let us choose $\eta_1, \eta_2 \in H, y_0 < \eta_1 < \eta_2 < |L_{z_0}|$. We have $x_0 \in A(\eta_2)$. Therefore, $x_0 \in A(\eta_1)$. Therefore, there exist sequences $\{\xi_n\}, \{x_n\}$, of points of $A(\eta_1)$ such that $\xi_n < x_n < x_0$, $\lim x_n = \lim x_n = x_0$. Property (c) follows from the fact that $x_n + iy_0$ and $x_n + iy_0$ lie in $\Sigma$.

4. Let $\Omega$ be the region \{z \mid Re z > 0\} - $\Sigma$, $(z = x + iy)$, where $\Sigma$ is the set found in §3. By the Riemann mapping theorem there exists a schlicht function, $w = f(z), f(\infty) = \infty$, mapping $\Omega$ onto the upper half $w$-plane. Since the limiting values of $f$ on the linear open set $R - A$ of the $x$-axis are real, it follows, by the strong form of the Schwarz reflection principle, that there is an extension of $f$ to a function schlicht in $\Omega_1 = \Omega \cup \Omega^* \cup \{z \mid Re z = 0, \forall z \in R - A\} \cup \{\infty\}$, with $f(\infty) = \infty, f(z^*) = f(z)^*$. (* denotes reflection in the real axis.) Evidently, $E_1 = \partial \Omega_1 = \Sigma \cup \Sigma^*$. The set of limiting values, $E_2$, of $f$ on $E_1$ is real, and according to Carathéodory's prime end theory [3], if $\Omega_5 = f(\Omega_1)$, then $E_2 = \partial \Omega_5$. In view of the reflection principle, $E_2$ is also the set of limiting values on $\Sigma$ of the restriction of $f$ to $\Omega$. By property (c), §3, each segment $L_x, x \in A$, with $|L_x| > 0$, is the impression of a prime end (of the second type) of $\partial \Omega$. The remaining points of $\Sigma$ (all
on the $x$-axis) are accessible points of $\partial \Omega$. Furthermore, since $A$ is totally disconnected, there are accessible points of $\partial \Omega$ in $R - A$ between any two prime ends of $\Sigma$. Hence, by Carathéodory's theory, $E_2$ is a totally disconnected set. This completes the proof of Theorem 1.

The fact that $E_2$ is totally disconnected makes it possible to interpret $E_2$ not only as the boundary of a (parallel) slit domain, but also, for instance, as the boundary of a circular slit domain. For example, by deleting $E_2$ along a radius from an annulus (or disk) one obtains an example of a circular slit annulus (or disk) which can be shown to be not minimal in the sense of [4] or [6].

**References**


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