ON THE PROXIMAL RELATION IN
TOPOLOGICAL DYNAMICS

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Let $\langle X, T \rangle$ be a transformation group with compact Hausdorff phase space $X$. The points $x$ and $y$ of $X$ are said to be proximal provided, whenever $\beta$ is a member of the unique compatible uniformity of $X$, there exists $t \in T$ such that $(xt, yt) \in \beta$. If $x$ and $y$ are not proximal, they are said to be distal.

Let $P$ denote the proximal relation in $X$. $P$ is a reflexive, symmetric, $T$ invariant relation, but is not in general transitive or closed. As is customary, if $x \in X$, let $P(x) = \{y \in X \mid (x, y) \in P\}$.

If $x \in X$, the orbit of $x$ is the set $xT = \{xt \mid t \in T\}$. The closure of $xT$, denoted by $(xT)^-$, is called orbit closure of $x$. A nonempty subset $M$ of $X$ is said to be a minimal orbit closure, or minimal set, if $M = (xT)^-$ for all $x \in M$. If $A$ is a nonempty closed, $T$ invariant subset of $X$, then $A$ contains at least one minimal set [3, 2.22].

We may consider $T$ as a subset of $X^X$. (We identify two elements $t_1$ and $t_2$ of $T$ if $xt_1 = xt_2$ for all $x \in X$.) Let $E$ be the closure of $T$ in $X^X$. $E$ is a compact semigroup (but not a topological semigroup); it is called the enveloping semigroup of $(X, T)$.

The enveloping semigroup of a transformation group was defined in [2]. Its algebraic properties, and their connection with the recursive properties of the transformation group are studied in [1].

A nonempty subset $I$ of $E$ is called a right ideal in $E$ if $IE \subseteq I$. If $I$ contains no proper nonempty subsets which are also right ideals, $I$ is called a minimal right ideal.

In Lemma 1, we summarize some results from [1] which we shall repeatedly use in this paper.

**Lemma 1.** (i) If $K$ is a closed right ideal in $E$, $K$ contains a minimal right ideal $I$.

(ii) If $I$ is a minimal right ideal in $E$, then $I$ is closed.

(iii) If $I$ is a minimal right ideal in $E$, and $x \in X$, then $xI$ is a minimal set in $X$.

(iv) If $K$ is a nonempty closed set in $E$ such that $K^2 \subseteq K$, then $K$ contains an idempotent (i.e., an element $u$ such that $u^2 = u$).

Presented to the Society, January 29, 1960; received by the editors December 22, 1959.

1 This work was supported in part by a National Science Foundation Research Grant.
(v) If $I$ is a minimal right ideal in $E$, $u$ an idempotent in $I$, and $p \in I$, then $up = p$.

(vi) The points $x$ and $y$ of $X$ are proximal if and only if there exists a minimal right ideal $I$ in $E$ such that $xp = yp$ for all $p$ in $I$.

(vii) $P$ is an equivalence relation in $X$ if and only if $E$ contains exactly one minimal right ideal.

Observe that (i) tells us that $E$ always contains at least one minimal right ideal.

**Lemma 2.** Let $x \in X$, and let $M$ be a minimal set contained in $(xT)^\sim$. Then there exists a minimal right ideal $I$ in $E$ such that $M = xI$. Moreover, there is a point $y \in M$ such that $x$ and $y$ are proximal.

**Proof.** Let $F = \{p \in E \mid xp \in M\}$. If $p \in F$ and $q \in E$, then $xpq \in Mq \subseteq M$, so $pq \in F$. Therefore, $FE \subseteq F$. Also, since $M$ is closed in $X$, $F$ is closed in $E$. That is, $F$ is a closed right ideal in $E$. By Lemma 1 (i), $F$ contains a minimal right ideal $I$. Then $xI \subseteq xF \subseteq M$. By Lemma 1 (iii), $xI$ is a minimal set in $X$. Consequently, $xI = M$.

Now, let $u$ be an idempotent in $I$. Then $y = xu \in xI = M$, and $yu = xu^2 = xu$, so $x$ and $y$ are proximal.

**Theorem 1.** Suppose that $P$ is an equivalence relation in $X$. Then

(i) $X = \bigcup_a N_a$, where the $N_a$ are pairwise disjoint, $N_a T \subseteq N_a$, and each $N_a$ contains precisely one minimal set $M_a$.

(ii) If $x \in N_a$, then $x$ is proximal to a point $y \in M_a$.

(iii) If $P$ is closed in $XX$, then the sets $N_a$ are closed.

**Proof.** (i) Let $x \in X$. Since proximal is an equivalence relation, $E$ contains just one minimal right ideal by Lemma 1, (vii). Therefore, by Lemma 2, $(xT)^\sim$ contains just one minimal set.

Let $\{M_a\}$ be the class of minimal sets in $X$. Let

$$N_a = \{x \in X \mid (xT)^\sim \supseteq M_a\}.$$ 

It now follows that the $N_a$ are pairwise disjoint, and that their union is $X$. By definition, each $N_a$ contains just one minimal set, namely $M_a$. It is clear that $N_a T \subseteq N_a$.

(ii) This is an immediate consequence of Lemma 2, since $M_a \subseteq (xT)^\sim$ whenever $x \in N_a$.

(iii) Suppose that $P$ is closed. Let $\{x_n \mid n \in D\}$ be a net in $N_a$, and suppose $x_n \to x$. Then there exists $y_n \in M_a$ such that $(x_n, y_n) \in P$. By choosing an appropriate subnet if necessary, let $y_n \to y \in M_a$. Since $P$ is closed $(x, y) \in P$. Therefore $M_a = (yT)^\sim \subseteq (xT)^\sim$ and $x \in N_a$. Hence $N_a$ is closed.
The following example shows that if \( P \) is not closed the sets \( N_\alpha \) need not be closed. The space \( X \) is a compact subset of the plane. \( X \) is the union of the circles \( C_n, \ (n = 0, 1, 2, \ldots) \) defined by \( x^2 + (y - 1 + n/(n^2 + 1))^2 = (1 - n/(n^2 + 1))^2 \). A point \( Q \) on \( C_n \) will be given the coordinates \((\alpha, n)\) where \( \alpha \) is the angle between 0 and \( \pi \) formed by the positive \( x \) axis and the chord from the origin to \( Q \).

We define a homeomorphism \( f \) of \( X \) onto \( X \) by \((\alpha, n)f = (\alpha + (1/n) \sin \alpha, n)\), if \( n > 0 \). We define \((\alpha, 0)f = (\alpha, 0)\). The group \( T \) consists of the positive and negative powers of \( f \). It is easily shown that \( P \) is an equivalence relation in \( X \).

If \( 0 \leq \alpha < \pi \), let \( M_\alpha \) be the set consisting of the point \((\alpha, 0)\). Then the sets \( M_\alpha \) are the minimal sets of \( X \). If \( 0 < \alpha < \pi \), \( N_\alpha = M_\alpha \). However, \( N_0 = (0, 0) \cup C_1 \cup C_2 \cup \cdots \), which is not closed.

Let \( u \) and \( v \) be idempotents in \( E \). We write \( u \sim v \) if \( uv = u \) and \( vu = v \). It is easy to see that \( \sim \) is an equivalence relation. In [1] it is proved that if \( I_1 \) and \( I_2 \) are minimal right ideals in \( E \), and \( u \) is an idempotent in \( I_1 \), then there is precisely one idempotent \( u_2 \) in \( I_2 \) such that \( u_1 \sim u_2 \).

Let \((X, T)\) and \((Y, T)\) be transformation groups with the same phase group \( T \). A homomorphism of \((X, T)\) onto \((Y, T)\) is a continuous map \( \phi \) of \( X \) onto \( Y \) such that \((xt)\phi = (x\phi)t\) for all \( x \in X \) and all \( t \in T \).

Let \( E(X) \) and \( E(Y) \) be the enveloping semigroups of the transformation groups \((X, T)\) and \((Y, T)\) respectively. It is proved in [2] that if \( \phi \) is a homomorphism of \((X, T)\) onto \((Y, T)\), then the map \( \theta : E(X) \rightarrow E(Y) \) defined by \((x\phi)(p\theta) = (xp)\phi(x \in X, p \in E(X))\) is a continuous onto semigroup homomorphism.

Let \( ((X_i, T) \mid i \in \mathcal{I}) \) be a family of transformation groups, with the same phase group \( T \). Let \( X = \bigcup_{i \in \mathcal{I}} X_i \), and let \( x = (x_i) \mid i \in \mathcal{I}) \in X \). If \( t \in T \) define \( xt = (x_it \mid i \in \mathcal{I}) \). With this definition of \( xt \), \((X, T)\) is a transformation group.

**Theorem 2.** Let \( ((X_i, T) \mid i \in \mathcal{I}) \) be a family of transformation groups with compact Hausdorff phase space, and the same phase group \( T \). Let \( X = \bigcup_{i \in \mathcal{I}} X_i \). Then proximal is an equivalence relation in \( X \) if and only if proximal is an equivalence relation in each \( X_i \).

**Proof.** Suppose proximal is an equivalence relation in each \( X_i \). Let \( \pi_i \) be the projection of \( X \) onto \( X_i \). Then \( \pi_i \) is a homomorphism of \((X, T)\) onto \((X_i, T)\). Let \( \theta_i \) be the induced homomorphism of \( E(X) \) onto \( E(X_i) \).

Now suppose that \( I \) and \( I' \) are distinct minimal right ideals in \( E(X) \). Then there are idempotents \( u \in I \) and \( u' \in I' \) with \( u \sim u' \). Then \( u\theta_i \) and \( u'\theta_i \) are idempotents in \( I\theta_i \) and \( I'\theta_i \) respectively such that \( u\theta_i \sim u'\theta_i \). Since proximal is an equivalence relation in \( X_i \), \( I\theta_i = I'\theta_i \).
and therefore $u\theta_i = u'\theta_i$. Then, if $x \in X$, $(xu)\pi_i = (x\pi_i)(u\theta_i) = (x\pi_i)(u'\theta_i) = (xu')\pi_i$. This is true for every $i \in \mathcal{I}$. That is $xu = xu'$, and since $x$ is arbitrary $u = u'$. But then $I \cap I' \neq \emptyset$. Therefore, $E(X)$ has only one minimal right ideal, and proximal is an equivalence relation in $X$.

Now suppose proximal is an equivalence relation in $X$. Let $j \in \mathcal{I}$, and suppose proximal is not an equivalence relation in $X_j$. Let $I_j$ and $I'_j$ be distinct minimal right ideals in $E(X_j)$. Since $I_j$ and $I'_j$ are closed and disjoint, $I_j\theta_j^{-1}$ and $I'_j\theta_j^{-1}$ are closed disjoint right ideals in $E(X)$. Hence they contain minimal right ideals $I$ and $I'$. This is a contradiction.

Let $\mathcal{P}$ be a property of a transformation group such that

1. $\mathcal{P}$ is hereditary. (That is, if $(Y, T)$ has property $\mathcal{P}$, and $M$ is an invariant subset of $Y$, then $(M, T)$ has property $\mathcal{P}$.)

2. $\mathcal{P}$ is productive. (That is, if $((X_i, T) | i \in \mathcal{I})$ is a family of transformation groups, each of which has property $\mathcal{P}$, then $(X, X, T)$ has property $\mathcal{P}$.)

Then, if $(X, T)$ is any transformation group, there exists a smallest closed, $T$ invariant equivalence relation $R$ in $X$ such that the transformation group $(X/R, T)$ has property $\mathcal{P}$ [2, Remark 8].

Let $\mathcal{P}$ denote the property "proximal is an equivalence relation." Obviously $\mathcal{P}$ is hereditary, and Theorem 2 tells us that $\mathcal{P}$ is productive. Hence, we may always divide out appropriately so that in the quotient transformation group proximal is an equivalence relation.

If $p \in E$ and $z = (x, y) \in X \times X$, we define $zp = (xp, yp)$. With this definition, we may consider $p$ as a member of the enveloping semigroup of $X \times X$. If $R \subseteq X \times X$ and $H \subseteq E$ then $RH$ is the set $\{zp | z \in R, p \in H\}$, where $zp$ is defined as above.

The diagonal of $X \times X$, that is, the set $\{(x, x) | x \in X\}$ will be denoted by $\Delta$.

Observe that if $z = (x, y) \in X \times X$, then $z \in P$ if and only if $(zT)^{-1} \cap \Delta \neq \emptyset$.

**Theorem 3.** The following are equivalent.

(i) $P$ is an equivalence relation in $X$.

(ii) Every orbit closure in $(X \times X, T)$ contains precisely one minimal set.\(^2\)

(iii) $PE \subseteq P$.

(iv) $PL \subseteq P$, where $L = \{I | I \text{ a minimal right ideal in } E\}$.

(v) $PL \subseteq \Delta$.

**Proof.** We show that each statement in the theorem implies the following one and that (v) implies (i).

\(^2\) This condition was suggested to me by Professor Robert Ellis.
Proof. If (i) holds, then by Theorem 2, proximal is an equivalence relation in $X \times X$. By Theorem 1 (i) every orbit closure in $X \times X$ contains precisely one minimal set.

Suppose (ii) holds. Let $z = (x, y) \in P$, and let $q \in E$. We show that $zq \in P$. By the remark above, it is sufficient to show that $(zqT)^{-} \cap \Delta \neq \emptyset$. Let $M$ be a minimal set contained in $(zqT)^{-}$. Since $zq \in (zT)^{-}$, $(zqT)^{-} \subset (zT)^{-}$ and therefore $M \subset (zT)^{-}$. Now $z \in P$, so $(zT)^{-} \cap \Delta \neq \emptyset$.

Since $(zT)^{-} \cap \Delta$ is a nonempty closed $T$ invariant set, it contains a minimal set $M'$. By (ii) $(zT)^{-}$ contains just one minimal set, so $M = M'$. Therefore, $M' \subset (zqT)^{-}$ and $(zqT)^{-} \cap \Delta \neq \emptyset$.

That (iii) implies (iv) is obvious.

Suppose (iv) is true. Let $z = (x, y) \in P$ and let $q \in I$. Then $q$ is in some minimal right ideal $I$. By (iv), $zq = (xq, yq) \in P$. That is, $xq$ is proximal to $yq$. By Lemma 1 (vi), there is a minimal right ideal $I'$ in $E$ such that $xqr = yqr$, for all $r \in I'$. Now $qI'$ is a right ideal, and $qI' \subset IE \subset I$, so we must have $qI' = I$. Hence there is an $r \in I'$ such that $qr = q$. Therefore, $xq = yq$, and $zq = (xq, yq) \in \Delta$. Therefore, $PL \subset \Delta$.

Finally, suppose (v) holds. Let $(x, y) \in P$ and $(y, z) \in P$. It follows from (v) that $xq = zq$ for all $q \in L$, so $(x, z) \in P$.

Corollary 1. If $P$ is closed in $X \times X$, then $P$ is an equivalence relation.

Proof. Let $z = (x, y) \in P$, and let $q \in E$. Let $\{t_n \mid n \in D\}$ be a net in $T$ such that $t_n \to q$. Now $(xt_n, yt_n) \in P$, and since $P$ is closed, $zq = (xq, yq) \in P$. Therefore, (iii) of Theorem 3 is satisfied.

The example given above shows that the converse of Corollary 1 is not true. For example, the points $(\pi/2, n)$ and $(0, 0)$ ($n = 1, 2, \cdots$), are proximal. But $(\pi/2, n) \to (\pi/2, 0)$ as $n \to \infty$ and $(\pi/2, 0)$ is clearly not proximal to $(0, 0)$ since both of these points are mapped into themselves by $f$.

If $X$ is minimal under $T$, it is not known if $P$ must be closed when $P$ is an equivalence relation.

If $I$ is a minimal right ideal in $E$, let $J(I)$ denote the set of idempotents in $I$. Let

$$J = \bigcup \{J(I) \mid I \text{ a minimal right ideal}\}.$$  

The transformation group $(X, T)$ is said to be pointwise almost periodic if, for every $x \in X$, $(xT)^{-}$ is a minimal set.

Theorem 4. Suppose $(X, T)$ is pointwise almost periodic, and let $x \in X$. Then

(i) $P(x) = xJ$.  


(ii) Let $I$ be a minimal right ideal in $E$. The points $xJ(I)$ are mutually proximal. If $y$ is proximal to all $x' \in xJ(I)$, then $y \in xJ(I)$. (That is, the sets $xJ(I)$ are "maximal" sets of mutually proximal points.)

(iii) Let $I$ be a minimal right ideal in $E$. Let $q \in I$ such that $(x, xq) \in P$ for all $x \in X$. Then $q \in J(I)$.

Proof. (i) It is clear that $xJ \subseteq P(x)$. Now, suppose $y \in P(x)$. Then there is a minimal right ideal $I$ in $E$ such that $xr = yr$, for all $r \in I$. Let $F = \{r \in I \mid yr = y\}$. Since $(yT)^-\text{ is minimal, } (yT)^- = yI$, and $F \neq \emptyset$. Now $F$ is closed, and $F^2 \subseteq F$. Therefore, $F$ contains an idempotent $u$. That is, there is a $u \in J(I)$ such that $yu = y$. But $xu = yu = y$, so $y \in xJ(I)$.

(ii) Suppose $u, v \in J(I)$. By Lemma 1 (v), $uv = v$. Therefore, $xuv = xv = xuv = xuv$, so $(xu, xv) \in P$.

If $(y, x') \in P$, for all $x' \in xJ(I)$, then in particular $(y, x) \in P$. Therefore, by (i), there exists a minimal right ideal $I'$ in $E$ and a $u' \in J(I')$ such that $y = xu'$. Let $u \in J(I)$ such that $u \sim u'$. By hypothesis, $(xu', xu) = (y, xu) \in P$. Then, for some minimal right ideal $I''$ in $E$, $xu'\phi' = xu\phi'$ for all $\phi' \in I''$. Let $u'' \in J(I'')$ such that $u \sim u' \sim u''$. Then $xu'u'' = xu'u''$. But $u'u'' = u'$, and $uu'' = u$, so $y = xu' = xu \in xJ(I)$.

(iii) Note that for all $x \in X$, and all $u \in J(I)$, $(xu, xq) \in P$. For by hypothesis, $(xu, xuq) \in P$. By Lemma 1 (v), $uq = q$, so $xuq = xq$.

Now, fix $x \in X$ and let $y = xq$. Then $y$ is proximal to $xu$, for all $u \in J(I)$. Hence, by (ii) $xq = y = xv$, where $v \in J(I)$ and $v$ depends (apparently) on $x$.

Now, by [1, Lemma 2 (3)] there is a $w \in J(I)$ such that $qw = q$. Since $vw = w$, we have $xq = xqw = xvw = xw$. That is, for all $x \in X$, $xq = xw$, so $q = w \in J(I)$.

References