ON THE PROXIMAL RELATION IN TOPOLOGICAL DYNAMICS

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Let \((X, T)\) be a transformation group with compact Hausdorff phase space \(X\). The points \(x\) and \(y\) of \(X\) are said to be proximal provided, whenever \(\beta\) is a member of the unique compatible uniformity of \(X\), there exists \(t \in T\) such that \((xt, yt) \in \beta\). If \(x\) and \(y\) are not proximal, they are said to be distal.

Let \(P\) denote the proximal relation in \(X\). \(P\) is a reflexive, symmetric, \(T\) invariant relation, but is not in general transitive or closed. As is customary, if \(x \in X\), let \(P(x) = \{y \in X \mid (x, y) \in P\}\).

If \(x \in X\), the orbit of \(x\) is the set \(xT = \{xt \mid t \in T\}\). The closure of \(xT\), denoted by \((xT)^-\) is called orbit closure of \(x\). A nonempty subset \(M\) of \(X\) is said to be a minimal orbit closure, or minimal set, if \(M = (xT)^-\) for all \(x \in M\). If \(A\) is a nonempty closed, \(T\) invariant subset of \(X\), then \(A\) contains at least one minimal set [3, 2.22].

We may consider \(T\) as a subset of \(X^X\). (We identify two elements \(t_1\) and \(t_2\) of \(T\) if \(xt_1 = xt_2\) for all \(x \in X\).) Let \(E\) be the closure of \(T\) in \(X^X\). \(E\) is a compact semigroup (but not a topological semigroup); it is called the enveloping semigroup of \((X, T)\).

The enveloping semigroup of a transformation group was defined in [2]. Its algebraic properties, and their connection with the recursive properties of the transformation group are studied in [1].

A nonempty subset \(I\) of \(E\) is called a right ideal in \(E\) if \(IE \subseteq I\). If \(I\) contains no proper nonempty subsets which are also right ideals, \(I\) is called a minimal right ideal.

In Lemma 1, we summarize some results from [1] which we shall repeatedly use in this paper.

**Lemma 1.** (i) If \(K\) is a closed right ideal in \(E\), \(K\) contains a minimal right ideal \(I\).

(ii) If \(I\) is a minimal right ideal in \(E\), then \(I\) is closed.

(iii) If \(I\) is a minimal right ideal in \(E\), and \(x \in X\), then \(xI\) is a minimal set in \(X\).

(iv) If \(K\) is a nonempty closed set in \(E\) such that \(K^2 \subseteq K\), then \(K\) contains an idempotent (i.e., an element \(u\) such that \(u^2 = u\)).

Presented to the Society, January 29, 1960; received by the editors December 22, 1959.

1 This work was supported in part by a National Science Foundation Research Grant.
(v) If $I$ is a minimal right ideal in $E$, $u$ an idempotent in $I$, and $p \in I$, then $up = p$.

(vi) The points $x$ and $y$ of $X$ are proximal if and only if there exists a minimal right ideal $I$ in $E$ such that $xp = yp$ for all $p$ in $I$.

(vii) $P$ is an equivalence relation in $X$ if and only if $E$ contains exactly one minimal right ideal.

Observe that (i) tells us that $E$ always contains at least one minimal right ideal.

**Lemma 2.** Let $x \in X$, and let $M$ be a minimal set contained in $(xT)^-$. Then there exists a minimal right ideal $I$ in $E$ such that $M = xI$. Moreover, there is a point $y \in M$ such that $x$ and $y$ are proximal.

**Proof.** Let $F = \{p \in E \mid xp \in M\}$. If $p \in F$ and $q \in E$, then $xpq \in Mq \subseteq M$, so $pq \in F$. Therefore, $FE \subseteq F$. Also, since $M$ is closed in $X$, $F$ is closed in $E$. That is, $F$ is a closed right ideal in $E$. By Lemma 1 (i), $F$ contains a minimal right ideal $I$. Then $xI \subseteq xF \subseteq M$. By Lemma 1 (iii), $xI$ is a minimal set in $X$. Consequently, $xI = M$.

Now, let $u$ be an idempotent in $I$. Then $y = xu \in xI = M$, and $yu = xu^2 = xu$, so $x$ and $y$ are proximal.

**Theorem 1.** Suppose that $P$ is an equivalence relation in $X$. Then

(i) $X = \bigcup a Na$, where the $Na$ are pairwise disjoint, $NaT \subseteq Na$, and each $Na$ contains precisely one minimal set $Ma$.

(ii) If $x \in Na$, then $x$ is proximal to a point $y \in Ma$.

(iii) If $P$ is closed in $XXX$, then the sets $Na$ are closed.

**Proof.** (i) Let $x \in X$. Since proximal is an equivalence relation, $E$ contains just one minimal right ideal by Lemma 1, (vii). Therefore, by Lemma 2, $(xT)^-$ contains just one minimal set.

Let $\{Ma\}$ be the class of minimal sets in $X$. Let

$$Na = \{x \in X \mid (xT)^- \supseteq Ma\}.$$ 

It now follows that the $Na$ are pairwise disjoint, and that their union is $X$. By definition, each $Na$ contains just one minimal set, namely $Ma$. It is clear that $NaT \subseteq Na$.

(ii) This is an immediate consequence of Lemma 2, since $Ma \subseteq (xT)^-$ whenever $x \in Na$.

(iii) Suppose that $P$ is closed. Let $\{x_n \mid n \in D\}$ be a net in $Na$, and suppose $x_n \to x$. Then there exists $y_n \in Ma$ such that $(x_n, y_n) \in P$. By choosing an appropriate subnet if necessary, let $y_n \to y \in Ma$. Since $P$ is closed $(x, y) \in P$. Therefore $Ma = (yT)^- \subseteq (xT)^-$ and $x \in Na$. Hence $Na$ is closed.
The following example shows that if $P$ is not closed the sets $N_\alpha$ need not be closed. The space $X$ is a compact subset of the plane. $X$ is the union of the circles $C_n$, $(n = 0, 1, 2, \ldots)$ defined by $x^2 + \left( y - 1 + n/(n^2 + 1) \right)^2 = \left( 1 - n/(n^2 + 1) \right)^2$. A point $Q$ on $C_n$ will be given the coordinates $(\alpha, n)$ where $\alpha$ is the angle between $0$ and $\pi$ formed by the positive $x$ axis and the chord from the origin to $Q$.

We define a homeomorphism $f$ of $X$ onto $X$ by $(\alpha, n)f = (\alpha + (1/n) \sin \alpha, n)$, if $n > 0$. We define $(\alpha, 0)f = (\alpha, 0)$. The group $T$ consists of the positive and negative powers of $f$. It is easily shown that $P$ is an equivalence relation in $X$.

If $0 \leq \alpha < \pi$, let $M_\alpha$ be the set consisting of the point $(\alpha, 0)$. Then the sets $M_\alpha$ are the minimal sets of $X$. If $0 < \alpha < \pi$, $N_\alpha = M_\alpha$. However, $N_0 = (0, 0) \cup C_1 \cup C_2 \cup \cdots$, which is not closed.

Let $u$ and $v$ be idempotents in $E$. We write $u \sim v$ if $uv = u$ and $vu = v$. It is easy to see that $\sim$ is an equivalence relation. In [1] it is proved that if $I_1$ and $I_2$ are minimal right ideals in $E$, and $u$ is an idempotent in $I_1$, then there is precisely one idempotent $u_2$ in $I_2$ such that $u_2 \sim u_2$.

Let $(X, T)$ and $(Y, T)$ be transformation groups with the same phase group $T$. A homomorphism of $(X, T)$ onto $(Y, T)$ is a continuous map $\phi$ of $X$ onto $Y$ such that $\phi(\pi(x)) = \pi(\phi(x))$ for all $x \in X$ and $\pi \in T$.

Let $E(X)$ and $E(Y)$ be the enveloping semigroups of the transformation groups $(X, T)$ and $(Y, T)$ respectively. It is proved in [2] that if $\phi$ is a homomorphism of $(X, T)$ onto $(Y, T)$, then the map $\theta : E(X) \to E(Y)$ defined by $(x\phi)(\theta) = (\phi_x)(x \in X, \theta \in E(X))$ is a continuous onto semigroup homomorphism.

Let $((X_i, T) \mid i \in S)$ be a family of transformation groups, with the same phase group $T$. Let $X = \bigcup_{i \in S} X_i$, and let $x = (x_i \mid i \in S) \in X$. If $t \in T$ define $xt = (x_i t_i \mid i \in S)$.

**Theorem 2.** Let $((X_i, T) \mid i \in S)$ be a family of transformation groups with compact Hausdorff phase space, and the same phase group $T$. Let $X = \bigcup_{i \in S} X_i$. Then proximal is an equivalence relation in $X$ if and only if proximal is an equivalence relation in each $X_i$.

**Proof.** Suppose proximal is an equivalence relation in each $X_i$. Let $\pi_i$ be the projection of $X$ onto $X_i$. Then $\pi_i$ is a homomorphism of $(X, T)$ onto $(X_i, T)$. Let $\theta_i$ be the induced homomorphism of $E(X)$ onto $E(X_i)$.

Now suppose that $I$ and $I'$ are distinct minimal right ideals in $E(X)$. Then there are idempotents $u \in I$ and $u' \in I'$ with $u \sim u'$. Then $u\theta_i$ and $u'\theta_i$ are idempotents in $I\theta_i$ and $I'\theta_i$ respectively such that $u\theta_i \sim u'\theta_i$. Since proximal is an equivalence relation in $X_i$, $I\theta_i = I'\theta_i$. 

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and therefore \( u\theta_i = u'\theta_i \). Then, if \( x \in X \), \((xu)\pi_i = (x\pi_i)(u\theta_i) = (x\pi_i)(u'\theta_i) = (xu')\pi_i \). This is true for every \( i \in \mathcal{I} \). That is \( xu = xu' \), and since \( x \) is arbitrary \( u = u' \). But then \( I \cap I' \neq \emptyset \). Therefore, \( E(X) \) has only one minimal right ideal, and proximal is an equivalence relation in \( X \).

Now suppose proximal is an equivalence relation in \( X \). Let \( j \in \mathcal{I} \), and suppose proximal is not an equivalence relation in \( X_j \). Let \( I_j \) and \( I'_j \) be distinct minimal right ideals in \( E(X_j) \). Since \( I_j \) and \( I'_j \) are closed and disjoint, \( I_j \theta_j^{-1} \) and \( I'_j \theta_j^{-1} \) are closed disjoint right ideals in \( E(X) \). Hence they contain minimal right ideals \( I \) and \( I' \). This is a contradiction.

Let \( \varphi \) be a property of a transformation group such that

1. \( \varphi \) is hereditary. (That is, if \( (Y, T) \) has property \( \varphi \), and \( M \) is an invariant subset of \( Y \), then \( (M, T) \) has property \( \varphi \).)

2. \( \varphi \) is productive. (That is, if \( ((X_i, T) \mid i \in \mathcal{I}) \) is a family of transformation groups, each of which has property \( \varphi \), then \( (X, \{X_i \mid i \in \mathcal{I}\}, T) \) has property \( \varphi \).)

Then, if \( (X, T) \) is any transformation group, there exists a smallest closed, \( T \) invariant equivalence relation \( R \) in \( X \) such that the transformation group \( (X/R, T) \) has property \( \varphi \) [2, Remark 8].

Let \( \varphi \) denote the property "proximal is an equivalence relation." Obviously \( \varphi \) is hereditary, and Theorem 2 tells us that \( \varphi \) is productive. Hence, we may always divide out appropriately so that in the quotient transformation group proximal is an equivalence relation.

If \( p \in E \) and \( z = (x, y) \in X \times X \), we define \( z p = (xp, yp) \). With this definition, we may consider \( p \) as a member of the enveloping semigroup of \( X \times X \). If \( R \subseteq X \times X \) and \( H \subseteq E \) then \( RH \) is the set \( \{zp \mid z \in R, p \in H\} \), where \( zp \) is defined as above.

The diagonal of \( X \times X \), that is, the set \( \{(x, x) \mid x \in X\} \) will be denoted by \( \Delta \).

Observe that if \( z = (x, y) \in X \times X \), then \( z \in P \) if and only if \( (zT)^{-1} \cap \Delta \neq \emptyset \).

**Theorem 3.** The following are equivalent.

(i) \( P \) is an equivalence relation in \( X \).

(ii) Every orbit closure in \( (X \times X, T) \) contains precisely one minimal set.*

(iii) \( PE \subseteq P \).

(iv) \( PL \subseteq P \), where \( L = \{I \mid I \) a minimal right ideal in \( E\} \).

(v) \( PL \subseteq \Delta \).

**Proof.** We show that each statement in the theorem implies the following one and that (v) implies (i).

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* This condition was suggested to me by Professor Robert Ellis.
Proof. If (i) holds, then by Theorem 2, proximal is an equivalence relation in $X \times X$. By Theorem 1 (i) every orbit closure in $X \times X$ contains precisely one minimal set.

Suppose (ii) holds. Let $z = (x, y) \in P$, and let $q \in E$. We show that $zq \in P$. By the remark above, it is sufficient to show that $(zqT)^- \cap \Delta \neq \emptyset$. Let $M$ be a minimal set contained in $(zqT)^-$. Since $zq \in (zT)^-$, $(zT)^- \subseteq (zT)^-$ and therefore $M \subseteq (zT)^-$. Now $z \in P$, so $(zT)^- \cap \Delta \neq \emptyset$. Since $(zT)^- \cap \Delta$ is a nonempty closed $T$ invariant set, it contains a minimal set $M'$. By (ii) $(zT)^-$ contains just one minimal set, so $M = M'$. Therefore, $M' \subseteq (zqT)^-$ and $(zqT)^- \cap \Delta \neq \emptyset$.

That (iii) implies (iv) is obvious.

Suppose (iv) is true. Let $z = (x, y) \in P$ and let $q \in L$. Then $q$ is in some minimal right ideal $I$. By (iv), $zq = (xq, yq) \in P$. That is, $xq$ is proximal to $yq$. By Lemma 1 (vi), there is a minimal right ideal $I'$ in $E$ such that $xqr = yqr$, for all $r \in I'$. Now $qI'$ is a right ideal, and $qI' \subseteq IE \subseteq I$, so we must have $qI' = I$. Hence there is an $r \in I'$ such that $qr = q$. Therefore, $xq = yq$, and $zq = (xq, yq) \in \Delta$. Therefore, $PL \subseteq \Delta$.

Finally, suppose (v) holds. Let $(x, y) \in P$ and $(y, z) \in P$. It follows from (v) that $xq = zq$ for all $q \in L$, so $(x, z) \in P$.

Corollary 1. If $P$ is closed in $X \times X$, then $P$ is an equivalence relation.

Proof. Let $z = (x, y) \in P$, and let $q \in E$. Let $\{t_n \mid n \in D\}$ be a net in $T$ such that $t_n \to q$. Now $(xt_n, yt_n) \in P$, and since $P$ is closed, $zq = (xq, yq) \in P$. Therefore, (iii) of Theorem 3 is satisfied.

The example given above shows that the converse of Corollary 1 is not true. For example, the points $(\pi/2, n)$ and $(0, 0)$ ($n = 1, 2, \cdots$), are proximal. But $(\pi/2, n) \to (\pi/2, 0)$ as $n \to \infty$ and $(\pi/2, 0)$ is clearly not proximal to $(0, 0)$ since both of these points are mapped into themselves by $f$.

If $X$ is minimal under $T$, it is not known if $P$ must be closed when $P$ is an equivalence relation.

If $I$ is a minimal right ideal in $E$, let $J(I)$ denote the set of idempotents in $I$. Let

$$J = U \{J(I) \mid I \text{ a minimal right ideal}\}.$$ 

The transformation group $(X, T)$ is said to be pointwise almost periodic if, for every $x \in X$, $(xT)^-$ is a minimal set.

Theorem 4. Suppose $(X, T)$ is pointwise almost periodic, and let $x \in X$. Then

(i) $P(x) = xJ$.
(ii) Let $I$ be a minimal right ideal in $E$. The points $xJ(I)$ are mutually proximal. If $y$ is proximal to all $x' \in xJ(I)$, then $y \in xJ(I)$. (That is, the sets $xJ(I)$ are "maximal" sets of mutually proximal points.)

(iii) Let $I$ be a minimal right ideal in $E$. Let $q \in I$ such that $(x, xq) \in P$ for all $x \in X$. Then $q \in J(I)$.

Proof. (i) It is clear that $xJ \subset P(x)$. Now, suppose $y \in P(x)$. Then there is a minimal right ideal $I$ in $E$ such that $xr = yr$, for all $r \in I$. Let $F = \{r \in I | yr = y\}$. Since $(yT)^- \subset F$ is minimal, $(yT)^- = yI$, and $F \neq \emptyset$. Now $F$ is closed, and $F^2 \subset F$. Therefore, $F$ contains an idempotent $u$. That is, there is a $u \in J(I)$ such that $yu = y$. But $xu = yu = y$, so $y \in xJ(I)$.

(ii) Suppose $u, v \in J(I)$. By Lemma 1 (v), $uv \in v$. Therefore, $xuv = xv = xuv^2 = xuv$, so $(xu, xv) \in P$.

If $(y, x') \in P$, for all $x' \in xJ(I)$, then in particular $(y, x') \in P$. Therefore, by (i), there exists a minimal right ideal $I'$ in $E$ and a $u' \in J(I')$ such that $y = xu'$. Let $u \in J(I)$ such that $u \sim u'$. By hypothesis, $(xu', xu) \in P$. Then, for some minimal right ideal $I''$ in $E$, $xv'u'' = xv'u''$ for all $u'' \in I''$. Let $u'' \in J(I'')$ such that $u \sim u'' \sim u'$. Then $xu'u'' = xuv'u''$. But $u'u'' = u'$, and $uu'' = u$, so $y = xu' = xu \in xJ(I)$.

(iii) Note that for all $x \in X$, and all $u \in J(I)$, $(xu, xq) \in P$. For by hypothesis, $(xu, xuq) \in P$. By Lemma 1 (v), $uq = q$, so $xuq = xq$.

Now, fix $x \in X$ and let $y = xq$. Then $y$ is proximal to $xu$, for all $u \in J(I)$. Hence, by (ii) $xq = y = xv$, where $v \in J(I)$ and $v$ depends (apparently) on $x$.

Now, by [1, Lemma 2 (3)] there is a $w \in J(I)$ such that $qw = q$. Since $vw = w$, we have $xq = xqw = xvw = xw$. That is, for all $x \in X$, $xq = xw$, so $q = w \in J(I)$.

References


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