be given \((r \geq s; (a_1', \ldots, a_s')\) may be the null set). Then (2.4) and (2.5) associate with (7.1)
\[
(7.2) \quad (p_1', \ldots, p_n') \quad \text{and} \quad (p_1', \ldots, p_n')
\]
respectively. Since the primary set (7.1) are distinct, there must be an \(a_i\) such that \(a_i \in \{a_1', \ldots, a_s'\}\). The \(n\)-tuplets (7.2) will be distinct since \(p_{a_i} \neq p_{a_i}'\). Therefore the lattice points associated with the primary sets (7.1) will be distinct.

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SUBGROUPS OF THE UNIMODULAR GROUP

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Following the notation of [3], we let \(\Gamma\) denote the proper unimodular group consisting of all 2\(\times\)2 matrices with rational integral elements and determinant +1. For \(m\) a positive integer, define the principal congruence group \(\Gamma(m)\) by
\[
(1) \quad \Gamma(m) = \{X \in \Gamma: X \equiv I \pmod{m}\},
\]
where \(I\) denotes the identity matrix in \(\Gamma\), and where congruence of matrices is interpreted as elementwise congruence.

For \(p\) a prime, we know from [2] that \(\Gamma(p)\) is a free group with a finite set \(S\) of generators. If we define
\[
(2) \quad T_m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix},
\]
then \(S\) may be chosen to include \(T_p\). For each fixed integer \(s\), we may define a group \(\Omega(p, s)\) consisting of all power products of the generators in \(S\) for which the exponent sum for each generator is a multiple of \(s\). In [3] it was shown that each \(\Omega(p, s)\) is a normal subgroup of \(\Gamma\) of finite index in \(\Gamma\). Furthermore, if \(s > 1\) and \((s, p) = 1\), it was proved that \(\Omega(p, s)\) does not contain any principal congruence group.

Let \(\Delta(m)\) denote the normal subgroup of \(\Gamma\) which is generated by \(T_m\). Obviously \(\Delta(m) \subseteq \Gamma(m)\). Recently, Brenner [1] raised the following questions:

A. Does \(\Delta(m) = \Gamma(m)\) for all \(m\)?

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B. For each $m$ does there exist a positive integer $k$ such that $\Delta(m) \supset \Gamma(mk)$?

The purpose of this note is to show that the answer to both questions is "No."

**Theorem.** If $m > 1$ and $m$ is not a power of a prime, then $\Delta(m)$ does not contain any principal congruence group.

**Proof.** From the hypothesis we may write

$$m = p^r s, \quad r \geq 1, \quad s > 1, \quad (p, s) = 1.$$  

Since $T_m$ is a power of $T_{ps}$, we have $\Delta(m) \subset \Delta(ps)$. Further, $T_{ps} = T_p^s$ implies that $T_{ps} \subseteq \Omega(p, s)$. But $\Omega(p, s)$ is a normal subgroup of $\Gamma$, and so we conclude

$$\Delta(m) \subset \Delta(ps) \subset \Omega(p, s).$$  

Since $\Omega(p, s)$ contains no principal congruence group, the same holds for $\Delta(m)$. Q.E.D.

Brenner showed that $\Delta(m) = \Gamma(m)$ for $1 \leq m \leq 5$. The above theorem implies that $\Delta(6)$ contains no principal congruence group. We are thus left with the following problem: For prime power values of $m$, can $\Delta(m)$ contain a principal congruence group?

The only additional light we can shed on this problem comes from Frasch [2], who showed:

Let $p \geq 7$, $p$ prime. Then $\Delta(p)$ is properly contained in $\Gamma(p)$.

**References**


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