ON THE KERNEL OF A TOPOLOGICAL SEMIGROUP WITH CUT POINTS

YATARÔ MATSUSHIMA

W. M. Faucett [1] recently studied the structure of the kernel of a compact connected mob which has a point that cuts the kernel. L. W. Anderson [2] has characterized the cut point of a connected topological lattice. The main purpose of this paper is to find a lattice theoretic characterization of the kernel by means of cut points in the topological semigroups derived from topological lattices. Using the concept of B-covers [3], we shall define a suitable multiplication in topological lattices to illustrate the structure of the kernel by lattice diagrams. The fact that a special case of Faucett's Theorem (Theorem 2) can be obtained from Anderson's result (Lemma 5) is important.

1. Preliminaries. We recall that a topological lattice is a Hausdorff space, \( L \), together with a pair of continuous functions \( \wedge : \mathbb{L} \times \mathbb{L} \rightarrow \mathbb{L} \) and \( \vee : \mathbb{L} \times \mathbb{L} \rightarrow \mathbb{L} \) which satisfy the usual conditions stipulated for a lattice. If \( X \) is a topological space and \( p \in X \), we say that \( p \) is a cut point of \( X \) if \( X \setminus p \) is not connected, i.e., if \( X \setminus p = U \cup V \) such that \( U \neq \emptyset \neq V \) and \( U^* \cap V = \emptyset = U \cap V^* \), where by \( A^* \) we mean the closure of \( A \).

Hereafter let \( S \) be a connected topological lattice which satisfies the modular law. Now we introduce a multiplication in \( S \) as follows:

\[ (M) \ xy = (a \vee x) \wedge (b \vee y) \]

for two fixed elements \( a, b \) of \( S \).

For any two elements \( a, b \) of a lattice \( S \) let

\[ B(a, b) = \{ x \mid (a \vee x) \wedge (b \vee x) = (a \wedge x) \vee (b \wedge x) = x \} \]

then \( B(a, b) \) is called the \( B \)-cover of \( a \) and \( b \) [3]. We define a mob to be a Hausdorff space together with a continuous associative multiplication. Then \( S \) is a mob with respect to \( (M) \), for the multiplication is continuous since \( S \) is a topological lattice and moreover it is associative by Lemma 1.

2. The kernel \( B(a, b) \) of a mob \( S \).

**Lemma 1.** \( x(yz) = (xy)z \) in \( S \).

**Proof.** We have
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\[ x(yz) = (a \lor x) \land (b \lor ((a \lor y) \land (b \lor z))) \]
\[ = (a \lor x) \land (a \lor b \lor y) \land (b \lor z), \]
\[ (xy)z = (a \lor ((a \lor x) \land (b \lor y))) \land (b \lor z) \]
\[ = (a \lor x) \land (a \lor b \lor y) \land (b \lor z) \] by the modular law.

**Lemma 2.** If \( x \in B(a, b), \ y \in S, \) then (i) \( xx = x, \) (ii) \( xy \in B(a, b), \)
\( yx \in B(a, b). \)

**Proof.** (i) follows from the definition of \( B(a, b). \)

(ii) \( (a \lor xy) \land (b \lor xy) \)
\[ = (a \lor ((a \lor x) \land (b \lor y))) \land (b \lor ((a \lor x) \land (b \lor y))) \]
\[ = (a \lor x) \land (a \lor b \lor y) \land (a \lor b \lor x) \land (b \lor y) \]
\[ = (a \lor x) \land (b \lor y) = xy \] by the modular law;
\[ (a \land xy) \lor (b \land xy) \]
\[ = (a \land ((a \lor x) \land (b \lor y))) \lor (b \land ((a \lor x) \land (b \lor y))) \]
\[ = (a \land (b \lor y)) \lor (b \land (a \lor x)) \]
\[ = (a \lor (b \land (a \lor x))) \land (b \lor y) \]
\[ = (a \lor b) \land (a \lor x) \land (b \lor y) \] by the modular law.

Since \( x \leq a \lor b \) for \( x \in B(a, b) \) we have \( (a \lor b) \land (a \lor x) \land (b \lor y) = (a \lor x) \land (b \lor y) = xy. \) Similarly we have \( yx \in B(a, b). \)

**Lemma 3.** Let \( p \in B(a, b); \) then \( Sp \) is a minimal left ideal and \( pS \) is a minimal right ideal.

**Proof.** We shall prove that \( (xp)(Sp) = xp \) for \( x \in S, \ p \in B(a, b). \)

For \( y \in S \) we have \( (xp)(yp) = (xpy)p = ((a \lor x) \land (a \lor b \lor p) \land (b \lor y))p \)
\[ = (a \lor x) \land (a \lor b \lor p) \land (a \lor b \lor y) \land (b \lor p) = (a \lor x) \land (b \lor p) = xp \] since \( p \leq a \lor b. \) Similarly we have \( (pS)(px) = px. \)

**Lemma 4.** If \( p \in B(a, b), \) then \( B(a, b) = SpS. \)

**Proof.** By Lemma 2 we have \( B(a, b) \supseteq SpS. \) If we take \( r \in pS \cap Sq \)
for \( q \in B(a, b), \) then we have \( qr = q \) by Lemma 3, where \( r = px \) for some \( x \in S. \) Accordingly we have \( B(a, b) \subseteq SpS. \)

As a consequence, we have the following theorem.

**Theorem 1.** \( B(a, b) \) is the kernel of a mob \( S. \)

3. The structure of the kernel \( B(a, b) \) with cut points.

**Lemma 5 (L. W. Anderson).** If \( S \) is a connected topological lattice and if \( p \in S \) then \( p \) is a cut point of \( S \) if, and only if, \( p \neq 0, p \neq I \) and \( L = (p \lor L) \cup (p \land L). \)
The next theorem is a special case of Faucett's theorem [1, Theorem 1.3].

**Theorem 2.** Let $S$ be a compact connected mob derived from a compact connected topological lattice introducing the multiplication $(M)$ into it. If there exists a point $p \in S$ that cuts $B(a, b)$, then we have either

(i) $B(a, b) = \{x \mid a \leq x \leq b\} = S_p$, that is, $B(a, b)$ is a minimal left ideal, and every element of $B(a, b)$ is left zero for $S$; or

(ii) $B(a, b) = \{x \mid b \leq x \leq a\} = pS$, that is, $B(a, b)$ is a minimal right ideal, and every element of $B(a, b)$ is right zero for $S$.

**Proof.** Since $p$ cuts $B(a, b)$, we have $B(a, b) = A \cup B$, where $A = \{x \mid x \leq p\}$ and $B = \{x \mid x \geq p\}$, by Lemma 5.

Now suppose that $a, b \leq p$; then for any element $x \in B$ such that $x > p$, we have $(a \land x) \lor (b \land x) = a \lor b \leq p < x$, that is, $x$ does not belong to $B(a, b)$, a contradiction. Similarly the case where $a, b \geq p$ does not occur. Thus we have either $a \leq p \leq b$ or $b \leq p \leq a$. In the first case, any element $x$ such that either $x < a$ or $b < x$ does not belong to $B(a, b)$. Now we shall prove that $B(a, b) = \{x \mid a \leq x \leq b\} = S_p$. Let $p \in B(a, b)$, $x \in S$; then $xp = (a \lor x) \land (b \lor x) = (a \lor x) \lor b = a \lor (b \land x)$ by the modular law. Then we have $a \leq xp \leq b$, hence $S_p \subset B(a, b)$.

Conversely, if we take $k \in B(a, b)$, then $kp = (a \lor k) \land (b \lor p) = k \land b = k$ since $a \leq k$, $p \leq b$. It follows that $B(a, b) \subset S_p$. Accordingly we have $S_p = B(a, b)$, and hence $B(a, b)$ is a minimal left ideal by Lemma 3.

Now let $x \in S$, $k \in B(a, b)$; then $kx = (a \lor k) \land (b \lor x) = k \land (b \lor x) = k$ since $k \leq b$, that is, every element of $B(a, b)$ is a left zero for $S$. This completes the proof of (i). Similarly we can prove (ii).

4. The case where no point cuts the kernel $B(a, b)$ for $S$. Throughout this section we shall assume that there is no point that cuts the kernel $B(a, b)$ of the mob $S$ derived from a topological lattice.

We can easily find that (i) if $a \leq b$, then $B(a, b)$ is a minimal left ideal for $S$, (ii) if $b \leq a$, then $B(a, b)$ is a minimal right ideal for $S$, (iii) if $a, b$ are noncomparable, then $B(a, b)$ has the same structure as that in Lemma 4.

Let us define a two-sided ideal $T$ of a mob $S$ to be a **prime ideal** provided that whenever $S \setminus T$ is non-null then $S \setminus \{T\}$ is a submob. A submob in a mob $S$ is a nonvoid set $T$ contained in $S$ such that $TT \subset T$. Now we shall find a necessary and sufficient condition for a two-sided ideal $C$ containing $B(a, b)$ to be a prime ideal in the case where $S \setminus g = C \cup D$, $C \neq \emptyset \neq D$ and $C^* \cap D = \emptyset = C \cap D^*$. In this case we do not assume that $S$ is connected.
Lemma 6. Let $S \setminus B(a, b) \ni z$; then $zz \in B(a, b)$ if, and only if, $z \leq a \uparrow b$.

Proof. By the modular law, we have $(a \uparrow zz) \cap (b \setminus zz) = zz$, $(a \setminus zz) \cap (b \setminus zz) = (a \uparrow z) \cap (b \setminus z) \cap (a \uparrow b)$. If $z \leq a \uparrow b$, then we have $zz \in B(a, b)$. Conversely if $zz \in B(a, b)$, then $zz = (a \uparrow b) \cap (a \uparrow z) \cap (b \setminus z) = (a \uparrow b) \setminus zz$, and hence $z \leq (a \uparrow z) \cap (b \setminus z) = zz \leq a \uparrow b$. Hence we have $z \leq a \uparrow b$.

Theorem 3. Let $S$ be a mob with respect to multiplication $(M)$, and let $z$ be an element of $S$ such that $S \setminus z = C \setminus D$, $C \neq D$, $C \neq D^*$, $C^* \cap D = \emptyset$, and $C$ is a two-sided ideal containing $B(a, b)$; then $C$ is a prime ideal if, and only if $z \geq a \uparrow b$.

Proof. By Lemma 5, we have either (i) $y < z < x$ or (ii) $y > z > x$ for all $x \in C$, $y \in D$. If $z \geq a \uparrow b$, let $S \setminus C = \{z, D\} = T \ni y_1, y_2$; then $y_1, y_2 \geq z \geq a \uparrow b$, and hence $y_1y_2 = (a \uparrow y_1) \cap (b \setminus y_2) = y_1 \setminus y_2 \geq z \geq a \uparrow b$, that is, $y_1y_2 \in T$. Then $C$ is a prime ideal. If $z > a \uparrow b$, then $z \leq a \uparrow b$ by Lemma 5. It follows that $zz \in B(a, b) \subset C$ by Lemma 6, and then $C$ is not a prime ideal. This completes the proof.

References


Gunma University, Maebashi, Japan