ON ISOMETRIC EQUIVALENCE OF CERTAIN
VOLterra OPERATORS

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The purpose of this paper is to extend the results of §4 of the
author's paper [3, referred to as V] to $L_p[0, 1]$ for all $p$ such that
$1 < p < \infty$. In general we shall use the notation and definitions of V,
except that the functions considered here are of the form

$$F(x, y) = (y - x)^{m-1}aG(x, y)$$

where

$$\begin{align*}
    m & \text{ is a positive integer,} \\
    |a| & = 1, \\
    G(x, x) & > 0;
\end{align*}$$

otherwise, as in V, the complex valued function $G(x, y)$ is continu-
ously differentiable. The only difference from V is the presence of the
constant $a$ which affects the proof of Theorem 2 of V. A version of
that theorem in the more general case where $a$ is an arbitrary con-
stant of absolute value 1 will be published elsewhere [4]. All other
theorems and proofs of V remain valid. The class $D$ of functions with
which we are principally concerned may be described as follows: the
functions $F$ are of the general form (1) where, in addition, $G$ and $m$
satisfy any one of the following: (A) $G$ is analytic in a suitable region
and $m$ is an arbitrary positive integer (see Lemma 4 of V); (B)
$G(x, y) = G(y - x)$ where $G(0) \neq 0$ and $G \in C^2$ in a neighborhood of
$y = x$ and otherwise $G(t) \in L_1[0, 1]$ and $m$ is an arbitrary positive
integer; (C) $G \in C^2$ and $m = 1$. One very important property of the
operators $T_F$ for $F \in D$ is the fact (see Theorem 3 of V) that their
only reducing manifolds are the subspaces $L_p[0, c]$ of $L_p[0, 1]$ for all
c $\in [0, 1]$ (see also [2; 5 and 6]). This property is crucial for the estab-
lishment of unitary invariants (in the case $p = 2$) of the operators
$T_F$ in §4 of V. As is usual, we define $q$ by $1/p + 1/q = 1$.

Two continuous linear transformations $T_1$ and $T_2$ mapping $L_p[0, 1]$
into itself are called isometrically equivalent if there exists an isometry
$U$ of $L_p[0, 1]$ onto itself such that $T_1 = UT_2U^{-1}$ (regarding isometries
for $p \neq 2$, see, e.g., [1, p. 178]; the considerations of the present paper
are valid without this restriction). Two preliminary lemmas are
needed in order to extend some results on Hilbert spaces and spectral
theory to general $p$. We shall use the following notation: $M_a$ is the
operator “multiplication by the characteristic function $\chi_a(x)$ of the

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interval \([0, a]\)”; similarly \(M_S\) is the operator “multiplication by the characteristic function \(c_S(x)\) of the subset \(S\) of \([0, 1]\).” We shall occasionally write \(E_a\) instead of \(M_a\).

**Lemma 1.** Let \(T_0\) be an idempotent bounded linear transformation of \(L_p[0, 1]\) into itself whose range is \(L_p[0, a]\) for some \(a \in (0, 1]\). Then \(\| T_0 \| = 1\) if and only if \(T_0 = M_a\).

**Proof.** Since \(\| M_a \| = 1\) is obvious, we turn to the converse. Let \(N_1 = L_p[0, a]\) and \(N_2 = L_p[a, 1]\). Then \(T_0 - M_a = T\) maps \(N_1\) into \(0\) and \(N_2\) into \(N_1\); we have \(T_0 = M_a + T\). We wish to show that \(T = 0\). Suppose that \(T \neq 0\); let \(b = \| T \| > 0\). Find a positive real number \(e\) such that

\[
(1 + b^q)^{1/q} - e \frac{b^{-1} + b^{q/p}}{(1 + b^q)^{1/p}} > 1;
\]

now determine \(f_2 \in N_2\) such that \(\| Tf_2 \|_p \geq (b - e)\| f_2 \|_p\) where

\[
\| f_2 \|_p = \frac{b^q}{1 + b^q}.
\]

Let \(f_1 = b^{-q}Tf_2 \in N_1\) and let \(f = f_1 + f_2\). Then \(T_0f = f_1 + Tf_2 = (b^{-q} + 1)Tf_2\).

Since \(f_i \in N_i\) \((i = 1, 2)\) we have \(\| f \|_p = \| f_1 \|_p + \| f_2 \|_p\) and

\[
\| f_1 \|_p = b^{-pa}b^p\| f_2 \|_p = \frac{b^{-pa}b^p}{1 + b^q} = \frac{1}{1 + b^q}.
\]

Therefore

\[
\| f \|_p = \| f_1 \|_p + \| f_2 \|_p \leq \frac{1}{1 + b^q} + \frac{b^q}{1 + b^q} = 1.
\]

On the other hand,

\[
\| T_0f \|_p = \| f_1 + Tf_2 \|_p = (b^{-q} + 1)\| Tf_2 \|_p \geq (b^{-q} + 1)(b - e)\| f_2 \|_p
\]

\[
= \frac{(b^{-q} + 1)(b - e)b^{q/p}}{(1 + b^q)^{1/p}}
\]

\[
= \frac{(b^{-q} + 1)b^q}{(1 + b^q)^{1/p}} - e \frac{b^{-1} + b^{q/p}}{(1 + b^q)^{1/p}}
\]

\[
= (1 + b^q)^{1/q} - e \frac{b^{-1} + b^{q/p}}{(1 + b^q)^{1/p}}
\]

\[
> 1
\]
by the choice of $e$, see (2). But this contradicts $||T_0|| = 1$ so that $b$ and hence $T$ must be zero, i.e., $T_0 = M_a$, and the proof of the lemma is complete.

The following lemma shows that the projections $E_a$ for all $a \in [0, 1]$ generate a maximal abelian algebra in the algebra of all bounded linear transformations of $L_p[0, 1]$ into itself not only for $p = 2$ but indeed for all $p$ considered in this paper.

**Lemma 2.** Let $T$ be a bounded linear transformation mapping $L_p[0, 1]$ into itself and suppose that $TE_a = E_aT$ for all $a \in [0, 1]$. Then there exists a bounded measurable function $f$ such that $T = M_f$ (= “multiplication by $f$”).

**Proof.** Let $e = e(x)$ be the function identically equal to 1 and let $f(x) = (Te)(x)$. We shall show that $f$ is essentially bounded and that $T = M_f$. If $g$ is a simple function: $g(x) = \sum_j a_j \varphi_j(x)$, then $Tg = \sum_j a_j T\varphi_j(x)$; but $\varphi_j(x) = (M_\varphi \varphi_j)(x)$ so that $(T\varphi_j)(x) = (TM_\varphi \varphi_j)(x) = (M_\varphi T\varphi_j)(x) = (M_\varphi f)(x)\varphi_j(x)$ since our hypothesis implies that $T$ commutes not only with all $E_a = M_a$ but also with all relevant $M_\varphi$. Therefore $(Tg)(x) = f(x)g(x)$ for all simple $g$. The boundedness of $T$ implies that $||Tg||_p \leq ||T||_p ||g||_p$, $\int_0^1 |f(x)g(x)|\, \rho dx \leq ||T||_p ||g||_p$ for all simple $g$. Hence $|f|^p$ and $|f|$ are essentially bounded and $(Tg)(x) = f(x)g(x)$ for all $g \in L_p[0, 1]$.

If $s = s(t)$ is a monotone increasing function defined on $[0, 1]$ such that $s(0) = 0$ and $s(1) = 1$, we write $U_s = M(s)$, where we use the notation of $V$. If $s(t)$ is absolutely continuous with an inverse function of the same kind, then $U_s$ as a linear transformation of $L_p[0, 1]$ into itself is an isometry onto.

**Theorem 1.** Let $T_{F_1}$ and $T_{F_2}$ be two continuous linear transformations of $L_p[0, 1]$ into itself whose only reducing manifolds are the subspaces $L_p[0, c]$ of $L_p[0, 1]$ for all $c \in [0, 1]$, such as, for example, the transformations $T_F$ for $F \in D$. Then if $T_{F_1}$ is isometrically equivalent to $T_{F_2} = UT_{F_1}U^{-1}$, there exist: (a) a measurable function $h(x)$ defined on $[0, 1]$ such that $h(x) = 1$; (b) a strictly monotone absolutely continuous function $s(x)$ defined on $[0, 1]$ such that $s(0) = 0$ and $s(1) = 1$ with an inverse function of the same kind. We have $U = M_hU_s$. The functions $F_1$ and $F_2$ are then related by the equation

$$F_2(x, y) = \frac{h(x)}{h(y)} (s'(x))^{1/p} (s'(y))^{1/q} F_1(s(x), s(y)).$$

If conversely two functions $F_1$ and $F_2$ are related by (3) where the functions $h(x)$ and $s(x)$ are defined as in (a) and (b) above, then $T_{F_1}$ is isometrically equivalent to $T_{F_2} = UT_{F_1}U^{-1}$ where $U = M_hU_s$. 

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Proof. Suppose first that $T_{F_1} = UT_{F_1}U^{-1}$. Since both linear transformations have as their only reducing manifolds the spaces $L_p[0, c]$ for all $c \in [0, 1]$, we can conclude that $UE_i U^{-1} = T_{F(t)}$ where $T_{F(t)}$ is idempotent with range $L_p[0, r(t)]$; therefore $r(t)$ is increasing and satisfies the equations $r(0) = 0$ and $r(1) = 1$. Since $\|E_i\| = 1$ and $U$ is an isometry, we have $\|T_{F(t)}\| = 1$ for all positive $r(t)$. Lemma 1 now implies that $T_{F(t)} = T_{E(t)} = UE_i U^{-1}$. To show that $r$ is absolutely continuous and strictly increasing, we consider for $g \in L_q[0, 1]$ the expression $(E_t o f, g) = (UE_i U^{-1} f, g) = (E_i f_1, g_1)$, where $f_1 = U^{-1} f$ and $g_1 = U^* g$; the linear transformation $U^*$ is the adjoint of $U$ (acting in $L_q[0, 1]$). If $f = g = 1$ then $r(t) = \int_0^t f_1(x) g_1(x) dx$. This shows that $r(t)$ is absolutely continuous. If $f_1 = g_1 = 1$ then $t = \int_0^r f(s) g(x) dx$. This shows that $r(t)$ is strictly increasing; the inverse function $s(t)$ of $r(t)$ has the same properties. It is easy to verify that $U^{-1} E_t U = E_t$; this equation together with the equation $UE_i U^{-1} = T_{E(t)}$ implies that $U_t U$ commutes with all $E_t$. Lemma 2 then implies that $U_i U = M_k$ where $k = k(x)$ is a bounded measurable function. Since $U_t U$ is an isometry, the function $k(x)$ must satisfy $|k(x)| = 1$; we have $U = U^{-1} M_k$. The functions $s(x)$ and $h(x) = k(s(x))$ are the functions whose existence was asserted by the theorem; a simple calculation shows that $U = U^{-1} M_k = M_h U_x$ as promised. It is now an easy matter to verify (3); the computation needed for this purpose is similar to that needed to establish the converse of the theorem.

We state next the analog of Theorem 5 of V; the formulas and proof are changed due to the presence of the constant $a$ in our present context, and to the arbitrariness of $p$.

Theorem 2. Let $F(x, y) = (y - x)^m a G(x, y)$ be of form (1) where $G \in C^1$ in a neighborhood of $y = x$ and let $T_F$, considered as a linear transformation of $L_p[0, 1]$ into itself, have as its only reducing manifolds the spaces $L_p[0, c]$ for all $c \in [0, 1]$, as is the case if $F \in D$. Then $T_F$ is isometrically equivalent to a unique $T_{F_1} = UT_{F_1}U^{-1}$ where $F_1$ and $G_1$ satisfy the following:

$$F_1(x, y) = (y - x)^m a G_1(x, y),$$

or

$$G_1(x, x) = c = \left( \int_0^1 (G(u, u))^{1/m} du \right)^m > 0,$$

for

$$\text{Im} (G_{1x}(x, x)) = \text{Im} (G_{1y}(x, x)) = 0.$$

This is achieved by setting $U = M_k U_x$, where

$$r(t) = (1/c)^{1/m} \int_0^t (G(u, u))^{1/m} du.$$
The function $h(x)$ is determined by defining

\[ F_0(x, y) = (y - x)^{m-1} a G_0(x, y) \]

by $T_{F_0} = U_r^{-1} T_F U_r$ where $G_0 = H_0 + iK_0$ for real $H_0$ and $K_0$ and setting $h(x) = \exp(-i/c) \int_0^x K_0(u, u) du$.

**Proof.** If $r$ and $h$ are defined as described and if we set $T_{F_1} = (M_1 U_r^{-1} T_F (M_1 U_r^{-1})^{-1}$ then $F_1$ does indeed satisfy (4). Thus every $T_F$ is isometric with $T_{F_1}$, where $F_1$ has form (4). To show uniqueness, suppose that $F_i = (y - x)^{m-1} a_i G_i$ ($i = 1, 2$) are both of form (4) and that $T_{F_1}$ is isometrically equivalent to $T_{F_2} = U T_{F_1} U^{-1}$. Then (3) implies that $F_1$ and $F_2$ are related by the following equation:

\[ (y - x)^{m-1} a G_0(x, y) \]

\[ = \frac{h(x)}{h(y)} (s'(x))^{1/p} (s'(y))^{1/q} \left( \frac{s(y) - s(x)}{y - x} \right)^{m-1} (y - x)^{m-1} a G_1(s(x), s(y)), \]

where the functions $h$ and $s$ are as in (a) and (b) of Theorem 2. On letting $y - x$ approach zero we see that $m_1 = m_2 = m$ and that

\[ a_2 G_2(x, x) = (s'(x))^{m_2} a_1 G_1(s(x), s(x)), \]  

since $1/p + 1/q + m - 1 = m$. Equation (5) implies that $a_1 = a_2 = a$ since $G_i(x, x) = c_i > 0$; we next observe that (5) also implies that $s(x) = x$ and that $a_1 = a_2 = a$. Equation (3) now reduces to $G_2(x, y) = (h(x)/h(y)) G_1(x, y)$ or

\[ h(y)(H_2(x, y) + iK_2(x, y)) = h(x)(H_1(x, y) + iK_1(x, y)), \]

if we write $G_j = H_j + iK_j$ for real $H_j$ and $K_j$ ($j = 1, 2$). Our hypotheses imply that $H_j(x, x) = c$ and that $K_j(x, x) = K_{ij}(x, x) = K_{jy}(x, x) = 0$ ($j = 1, 2$) and also that $h(x) = \exp(ik(x))$ for real $k(x)$ is differentiable. Differentiation of (6) and setting $x = y$ yields $h(x)H_{22}(x, x) = ch'(x) + h(x)H_{12}(x, x)$ so that $h'/h = ik' = 1/c(H_{22}(x, x) - H_{12}(x, x))$. But the last expression is real so that $k' = 0$, and $h$ is constant. We finally arrive at $G_1 = G_2$: If two functions $F_1$ and $F_2$ satisfy (4) and if the corresponding operators $T_{F_1}$ and $T_{F_2}$ are isometrically equivalent and have the spaces $L_p[0, c]$ for all $c \in [0, 1]$ as their only reducing manifolds—for example, if the functions $F_j \in \mathcal{D}$—then $F_1 = F_2$.

Observe that if our functions $F$ belong to $D$, then the similarity invariants of $T_F$, viz., $m, a$, and $c$, enter directly into the formulation of the isometry invariants (see V [4] for similarity invariants). The “canonical functions” $F_i$ as given by (4) are the same for all $p$; however a given $T_F$ will have as its “canonical form” $T_{F_0}$, a transformation which in general does depend on $p$. If, for example, $F(x, y)$
= 1 + 2x + i(x - y), then m = 1, a = 1, c = 2. To describe its "canonical form" $F_1$ satisfying (4), it is convenient to introduce the function $K(x, y) = \left(\frac{(8x + 1)(8y + 1)}{8y + 1}\right)^{1/2}$. A simple calculation shows that $F_1(x, y) = 2 \exp(-i \log K)(K^{1/2} + i(K^{1/2} - K^{-1/2}))$.

References


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THE NONEXISTENCE OF PROJECTIONS FROM $L^1$ TO $H^1$

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Consider the Banach space $L^1(0, 2\pi)$ and the subspace $H^1$, of all functions all of whose negative Fourier coefficients vanish. It has been conjectured that $H^1$ has no complement in $L^1$, i.e., that $L^1$ is not the direct sum of $H^1$ and some other Banach space. In this note we give a proof of this conjecture.

The conjecture is of course equivalent to the following statement on projection operators.

Theorem. There exists no bounded linear operator $P : L^1 \to H^1$ for which $Pf = f$ for all $f \in H^1$.

Proof. Suppose such a $P$ existed. Let $l_n(f)$ denote the $n$th Fourier coefficient of $P(f)$; then $l_n$ is a bounded linear functional on $L^1$ and as a result we have

$$l_n(f(\theta)) = \int_0^{2\pi} f(\theta)\phi_n(\theta)d\theta,$$

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