= 1 + 2x + i(x - y), then m = 1, a = 1, c = 2. To describe its “canonical form” $F_1$ satisfying (4), it is convenient to introduce the function $K(x, y) = ((8x + 1)/(8y + 1))^{1/2}$. A simple calculation shows that $F_1(x, y) = 2 \exp(-i \log K)(K^{1/2}v + i(K^{1/2} - K^{-1/2})).$

References


THE NONEXISTENCE OF PROJECTIONS FROM $L^1$ TO $H^1$

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Consider the Banach space $L^1(0, 2\pi)$ and the subspace $H^1$, of all functions all of whose negative Fourier coefficients vanish. It has been conjectured that $H^1$ has no complement in $L^1$, i.e., that $L^1$ is not the direct sum of $H^1$ and some other Banach space. In this note we give a proof of this conjecture.

The conjecture is of course equivalent to the following statement on projection operators.

**Theorem.** There exists no bounded linear operator $P: L^1 \to H^1$ for which $Pf = f$ for all $f \in H^1$.

**Proof.** Suppose such a $P$ existed. Let $l_n(f)$ denote the $n$th Fourier coefficient of $P(f)$; then $l_n$ is a bounded linear functional on $L^1$ and as a result we have

$$l_n(f(\theta)) = \int_0^{2\pi} f(\theta)\phi_n(\theta)d\theta,$$

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where $\phi_n(\theta)$ is a bounded function. Also we know that

$$l_n(e^{n\theta}) = 1 \quad (\text{assuming } n \geq 0)$$

and so

$$\int_{0}^{2\pi} |\phi_n(\theta)| \, d\theta \geq 1.$$

It follows that

$$\int_{0}^{2\pi} \sum_{n=1}^{N} \frac{|\phi_n(\theta)|}{n} \, d\theta \geq \log N, \quad N > 1, \quad N \text{ fixed}.$$

So that, for some $\theta_0$, $\sum_{n=1}^{N} |\phi_n(\theta_0)|/n \geq (1/2\pi) \log N$. Thus choosing $\epsilon_n = \text{sgn} \, \phi_n(\theta_0)$ we are assured that $\max_{\theta} \left| \sum_{n=1}^{N} \epsilon_n \phi_n(\theta)/n \right| \geq (1/2\pi) \log N$ (the inequality holding in fact for $\theta_0$).

We can therefore determine a function $f(\theta)$ with $\int_{0}^{2\pi} |f| \leq 1$ such that

$$\int_{0}^{2\pi} f(\theta) \sum_{n=1}^{N} \frac{\epsilon_n \phi_n(\theta)}{n} \, d\theta \geq \frac{1}{10} \log N,$$

but

$$\left| \int_{0}^{2\pi} f(\theta) \sum_{n=1}^{N} \frac{\epsilon_n \phi_n(\theta)}{n} \, d\theta \right| \leq \sum_{n=1}^{N} \frac{|l_n(f)|}{n} = \sum_{n=1}^{N} \frac{|a_n|}{n},$$

where $\sum a_n e^{in\theta}$ is the Fourier series for Pf. Since $Pf \in H^1$, however, by Hardy's theorem [1],

$$\sum_{n=1}^{N} \frac{|a_n|}{n} \leq 2\pi \int |Pf| \leq 2\pi \|P\|.$$

Finally combining (1) and (2) gives the contradiction

$$\frac{1}{10} \log N \leq 2\pi \|P\|,$$

for all $N > 1$,

and this completes the proof.

Reference