restricting the boundary points \( \alpha_1, \alpha_2 \) further than requiring them to belong to \([a, \alpha_2], [\alpha_2, b]\), respectively.

Conditions are not imposed on \( a_{11}(x), a_{nn}(x) \). If these functions are identically zero over \((\alpha_1, \alpha_2)\) Theorem 2 follows for weaker restrictions than (C), (D). For the \( a_{1f}(x), a_{ne}(x), f = 2, \ldots, n, e = 1, \ldots, n - 1 \), it is sufficient to require that the positive functions be nonnegative, the negative functions be nonpositive and \( a_{1n}(\alpha_2), a_{n1}(\alpha_2) > 0 \).

Auburn University

A MOORE SPACE ON WHICH EVERY REAL-VALUED CONTINUOUS FUNCTION IS CONSTANT

STEVE ARMENTROUT

F. B. Jones [2] recently gave an example of a Moore space \( \Lambda_\infty \) in which there exists a point \( x \) such that \( \Lambda_\infty \) is not completely regular at \( x \). It is easy to modify the construction used by Jones so as to obtain a Moore space \( A \) in which there exist distinct points \( a \) and \( b \) such that for every real-valued continuous function \( f \) on \( A, f(a) = f(b) \). Upon applying Urysohn's process of condensation of the singularities of the space \( A \) [4], in a manner similar to that used by Hewitt [1], there results a Moore space \( X \) on which every real-valued continuous function is constant.

Throughout this paper, \( J \) denotes the set of positive integers. A sequence is a function on \( J \), and if \( f \) is a sequence and \( n \in J \), then \( f_n \) denotes \( f(n) \).

By a Moore space is meant a topological space \( X \) whose topology has a basis consisting of sets termed regions, satisfying the following condition (axiom 13, that is, parts 1, 2, and 3 of axiom 1, of [3]): There exists a sequence \( G \) such that (1) if \( n \in J, G_n \) is a collection of regions covering \( X \), (2) if \( n \in J, G_{n+1} \subset G_n \), and (3) if \( r \) is a region, \( x \in r \), and \( y \in r \), then there exists a positive integer \( n \) such that if \( g \in G_n \) and \( x \in g \), then \( g \subset (r - \{x\}) \cup \{y\} \). The following characterization of a Moore space will be used in this paper: \( X \) is a Moore space if and only if \( X \) is a regular Hausdorff space for which there exists a sequence \( G \) of open coverings of \( X \) such that if \( U \) is an open set and

Presented to the Society, September 3, 1959; received by the editors February 2, 1960.
there exists a positive integer \( n \) such that if \( g \in G_n \) and \( p \in g \), then \( g \subseteq U \) [5].

1. **Construction of the space \( A \).** Jones, in his construction of the space \( A_\infty \), made use of a certain Moore space \( \Lambda \), an infinite sequence \( \Lambda_1, \Lambda_2, \Lambda_3, \cdots \) of disjoint spaces, each congruent with \( \Lambda \), and an ideal point \( p \) [2]. Adjacent terms of the sequence are pieced together along their boundaries in a certain way. The space \( A \) is to be constructed by using a doubly infinite sequence \( \cdots, \Lambda_{-2}, \Lambda_{-1}, \Lambda_0, \Lambda_1, \Lambda_2, \cdots \) of disjoint spaces, each congruent with \( \Lambda \), and two ideal points, \( a \) and \( b \). Adjacent terms of the sequence are pieced together along their boundaries as in the construction of \( A_\infty \). Neighborhoods of \( a \) are defined as those for \( p \) are in the case of \( A_\infty \), and neighborhoods of \( b \) are defined in an obvious manner. \( A \) is a Moore space of cardinal \( c \), and a slight modification of Jones' proof that \( A_\infty \) is not completely regular at \( p \) shows that if \( f \) is a continuous real-valued function on \( A \), then \( f(a) = f(b) \).

2. **Construction of the space \( X \).** Now there will be constructed a Moore space \( X \) on which every real-valued continuous function is constant. Consider a collection of \( c \) disjoint spaces, each homeomorphic with the space \( A \). This collection may be well-ordered as an order of ordinal number \( \Delta \), the initial ordinal of the cardinal \( c \); let \( A^1, A^2, A^3, \cdots, A^\lambda, \cdots \), \( \lambda < \Delta \), be one such order. If \( \lambda < \Delta \), then \( A^\lambda \) has cardinal \( c \) and may be well-ordered as an order of ordinal number \( \Delta \); let \( x_1^\lambda, x_2^\lambda, x_3^\lambda, \cdots, x_n^\lambda, \cdots, \beta_1 < \Delta, \beta_2 < \Delta, \beta_3 < \Delta, \cdots \), be one such order where \( x_1^\lambda \) is the point \( a \) of the space \( A^\lambda \) and \( x_2^\lambda \) is the point \( b \) of \( A^\lambda \).

Let \( Q \) be the set of all ordered quadruples \((\lambda_1, \beta_1; \lambda_2, \beta_2)\) where each of \( \lambda_1, \beta_1, \lambda_2, \beta_2 \) is an ordinal number less than \( \Delta \), and either \( \lambda_1 < \lambda_2 \), or \( \lambda_1 = \lambda_2 \) and \( \beta_1 < \beta_2 \). It can be shown by transfinite induction that there exists a one-to-one function \( \Gamma \) with domain \( Q \) and range the set of ordinals less than \( \Delta \) such that (1) \( \Gamma(1, 1; 1, 2) = 1 \) and \( \Gamma(\lambda_1, \beta_1; \lambda_2, \beta_2) > \lambda_2 \) for all elements of \( Q \) other than \( (1, 1; 1, 2) \). Let \( \phi \) be a function such that

\[
\phi[x_1^{\Gamma(\lambda_1, \beta_1; \lambda_2, \beta_2)}] = \lambda_1 \quad \text{and} \quad \phi[x_2^{\Gamma(\lambda_1, \beta_1; \lambda_2, \beta_2)}] = \lambda_2.
\]

The function \( \phi \) maps \( x_1^\lambda, 1 < \lambda < \Delta \), into \( A^\gamma \) for some \( \gamma \) less than \( \lambda \), and similarly for \( x_2^\lambda \).

For each ordinal \( \lambda \) and each ordinal \( \beta, \lambda < \Delta \) and \( 2 < \beta < \Delta \), \( x_\beta^\lambda \) is an initial point; \( x_1^\lambda \) and \( x_2^\lambda \) are also initial points.

Certain sequences of points of \( U_{\gamma < \Delta} A^\gamma \) are defined to be chains. \( C \) is a chain if and only if \( C \) is a sequence, \( x_{\beta_1}^\lambda, x_{\beta_2}^\lambda, x_{\beta_3}^\lambda, \cdots \), such that
\(\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \Delta\), \(x^\lambda_1\) is an initial point, and if \(n \in J\), then 
\[\phi(x^\lambda_{n+1}) = x^\lambda_n\]
Each chain contains one and only one initial point, and if \(x^\lambda_0\) is an initial point, then \(C^\lambda_0\) is the union of all chains with \(x^\lambda_0\) as an initial point. If \(x^\lambda_0\) and \(x^\lambda_1\) are two distinct initial points, then 
\(C^\lambda_0\) and \(C^\lambda_1\) are disjoint. Further, if \(i\) is either 1 or 2, and \(1 < \lambda < \Delta\), then there is only one set \(C^\alpha_\lambda\) such that \(x^\lambda_i\) belongs to a chain of \(C^\alpha_\lambda\). If \(x^\lambda_0\) belongs to a chain of \(C^\alpha_\lambda\), then \(x^\lambda_i\) is a co-ordinate of \(C^\alpha_\lambda\). Let \(X\) be the collection of all sets \(C^\alpha_\lambda\) for \(\lambda < \Delta\) and \(\beta < \Delta\).

Now a topology for \(X\) will be constructed so that the resulting space is a Moore space. Suppose that \(\lambda\) is an ordinal, \(\lambda < \Delta\). The space \(A^\lambda\) is a Moore space. Let \(G^\beta\) be a sequence of collections of regions of the space \(A^\lambda\) satisfying axiom 13 relative to \(A^\lambda\). Suppose now that \(\beta\) is an ordinal, \(\beta < \Delta\). Let \(g^\lambda_1\) be a sequence of regions such that (1) if \(n \in J\), then \(x^\lambda_n \in g^\lambda_n\) and \(g^\lambda_n \subseteq G^\lambda_n\), (2) if \(n \in J\), then \((g^\lambda_{n(n+1)})^{-1} \subseteq g^\lambda_{n+1}\) and (3) if \(2 < \beta\), then \(x^\lambda_1 \notin (g^\lambda_{1n})^{-1}\) and \(x^\lambda_2 \notin (g^\lambda_{2n})^{-1}\). The sequences \(g^\lambda_1\) and \(g^\lambda_2\) are to satisfy the additional condition that 
\[(g^\lambda_1)_{-1} \cap (g^\lambda_2)_{-1} = \emptyset\].

Regions for \(X\) will now be defined. Suppose that \(x \in X\). There exist an ordinal \(\lambda\), \(\lambda < \Delta\), and an ordinal \(\beta\), \(\beta < \Delta\), such that \(x = C^\alpha_\lambda\). Suppose that \(n \in J\). \(W^\lambda_{\beta n}\) is the set of all \(C^\alpha_\lambda\) which have initial points in \(g^\lambda_{\beta n}\). \(W^\lambda_{\beta n}\) is defined as follows: (a) If both \(x^\lambda_{1n}\) and \(x^\lambda_{2n}\) are co-ordinates of elements of \(W^\lambda_{\beta n}\), then \(W^\lambda_{\beta n}\) is the set of all \(C^\alpha_{n+1}\) with initial points in \((g^\lambda_{1n+1} \cup g^\lambda_{2n+1})^{-1}\). (b) If \(x^\lambda_{1n}\) is a co-ordinate of an element of \(W^\lambda_{\beta n}\) but \(x^\lambda_{2n}\) is not a co-ordinate of any element of \(W^\lambda_{\beta n}\), \(i, j = 1, 2; i \neq j\), then \(W^\lambda_{\beta n}\) is the set of all \(C^\alpha_{n+1}\) with initial point in \(g^\lambda_{1n+1}^{-1}\). (c) If neither \(x^\lambda_{1n}\) nor \(x^\lambda_{2n}\) is a co-ordinate of any element of \(W^\lambda_{\beta n}\), then \(W^\lambda_{\beta n} = \emptyset\).

Suppose that \(\nu\) is an ordinal, \(\lambda + \nu < \Delta\), and for each ordinal \(\mu\), \(\mu < \nu\), \(W^\mu_{\beta n}\) has been defined. Let \(Y^\lambda_{\beta n} = \cup_{\mu < \nu} W^\lambda_{\beta n}\). Then \(W^\lambda_{\beta n}\) is defined as follows: (a) If both \(x^\lambda_{1n}\) and \(x^\lambda_{2n}\) are co-ordinates of elements of \(Y^\lambda_{\beta n}\), then \(W^\lambda_{\beta n}\) is the set of all \(C^\alpha_{n+\nu}\) with initial point in the set \((g^\lambda_{1n+1} \cup g^\lambda_{2n+1})^{-1}\). (b) If \(x^\lambda_{1n}\) is a co-ordinate of an element of \(Y^\lambda_{\beta n}\) but \(x^\lambda_{2n}\) is not a co-ordinate of any element of \(Y^\lambda_{\beta n}\), \(i, j = 1, 2; i \neq j\), then \(W^\lambda_{\beta n}\) is the set of all \(C^\alpha_{n+\nu}\) with initial point in \(g^\lambda_{1n+1}^{-1}\). (c) If neither \(x^\lambda_{1n}\) nor \(x^\lambda_{2n}\) is a co-ordinate of any element of \(Y^\lambda_{\beta n}\), then \(W^\lambda_{\beta n} = \emptyset\).

Let \(w^\lambda_{\beta n} = \cap_{\delta \in \Delta} W^\lambda_{\beta \delta n}\). \(R\) is a region in \(X\) if and only if for some positive integer \(n\) and some element \(C^\Delta_\beta\) of \(X\), \(R\) is \(w^\lambda_{\beta n}\).

By making minor modifications in Hewitt’s proof [1], one may show that \(R\) is a regular Hausdorff space.

Suppose that \(n \in J\); let \(\mathcal{K}_n\) be the set of all \(w^\lambda_{\beta n}\) for all \(C^\beta_\delta\) belonging to \(X\). It is clear that if \(n \in J\), \(\mathcal{K}_n\) is an open cover of \(X\). It will now be shown that if \(\mathcal{U}\) is a region in \(X\) and \(p \in \mathcal{U}\), then there exists a
positive integer $n$ such that if $h \in \mathcal{C}_n$ and $p \in h$, then $h \subseteq \mathcal{U}$. From this it follows that $X$ is a Moore space.

Suppose that $\mathfrak{U}$ is a region; for some $\lambda$, $\beta$, and positive integer $n$, $\mathfrak{U} = \mathfrak{W}^\lambda_{\beta n}$. Suppose that $p \in \mathfrak{U}$; for some $\mu$ and $\alpha$, with $\lambda \leq \mu$, $p = C^\mu_{\alpha}$. There are two cases:

(1) $\lambda = \mu$. If $\lambda > 1$, then $\alpha > 2$ and there exists a positive integer $m$ such that if $j \in J$ and $j > m$, then $x^\mu_{\alpha} \in (g^\lambda_{\beta j} \cup g^\lambda_{\beta j'})$. If $\lambda = 1$, take $m$ to be 1. Now suppose that $k \in J$ and $k > m$. Suppose further that $h \in \mathcal{C}_k$ and $C^\lambda_{\alpha} \subseteq h$. Then for some $\gamma$, $h = \mathfrak{W}^\lambda_{\gamma k}$. For clearly $C^\lambda_{\alpha}$ does not belong to any $\mathfrak{W}^\sigma_{\alpha}$ for $\sigma > \lambda$. If there exists an ordinal $\sigma$, $\sigma < \lambda$, and an ordinal $\epsilon$ such that $C^\lambda_{\alpha} \subseteq \mathfrak{W}^\epsilon_{\beta n}$, then one of $x^\lambda_{1\alpha}$ and $x^\lambda_{2\alpha}$ is a co-ordinate of an element of $\mathfrak{W}^\epsilon_{\beta n}$, and $x^\lambda_{\alpha} \in (g^\lambda_{\beta j} \cup g^\lambda_{\beta j'})$. However, as $k > m$, $x^\lambda_{\alpha} \in (g^\lambda_{\beta j} \cup g^\lambda_{\beta j'})$.

Since $C^\lambda_{\alpha} \subseteq \mathfrak{W}^\lambda_{\beta n}$, then $x^\lambda_{\alpha} \in \mathfrak{W}^\lambda_{\beta n}$. As $A^\mu_{\alpha}$ is a Moore space, there exists a positive integer $q$ such that if $g \in C^\mu_{\alpha}$ and $x^\mu_{\alpha} \in g$, then $g \subseteq \mathfrak{W}^\gamma_{\gamma q}$; clearly $q$ may be taken so that $q > m$. Now suppose that $h \in \mathcal{C}_q$ and $C^\lambda_{\alpha} \subseteq h$. Then as $q > m$, for some $\gamma$, $h = \mathfrak{W}^\lambda_{\gamma q}$; since $x^\lambda_{\alpha} \in \mathfrak{W}^\lambda_{\gamma q}$, then $g^\lambda_{\gamma q} \subseteq \mathfrak{W}^\lambda_{\beta n}$ and hence $\mathfrak{W}^\gamma_{\gamma q} \subseteq \mathfrak{W}^\lambda_{\beta n}$. Thus if $h \in \mathcal{C}_n$ and $C^\lambda_{\alpha} \subseteq h$, then $h \subseteq \mathfrak{U}$.

(2) $\lambda < \mu$. In this case, $\alpha > 2$. Since $C^\mu_{\alpha} \subseteq \mathfrak{W}^\mu_{\beta n}$, then one of $x^\mu_{1\alpha}$ and $x^\mu_{2\alpha}$ is a co-ordinate of an element of $\mathfrak{W}^\mu_{\beta n}$, and $x^\mu_{\alpha}$ belongs to one of $g^\mu_{\beta n}$ and $g^\mu_{\beta n}$. Suppose $x^\mu_{i\alpha} \in g^\mu_{\beta n}$, $i = 1$ or 2.

There exists a positive integer $m$ such that if $j \in J$ and $j > m$, then $x^\mu_{i\alpha} \in (g^\mu_{\beta j} \cup g^\mu_{\beta j'})$. Then, as in case (1), if $k \in J$, $k > m$, $h \in \mathcal{C}_k$, and $C^\mu_{\alpha} \subseteq h$, then for some $\gamma$, $h = \mathfrak{W}^\mu_{\gamma k}$. As $A^\mu_{\alpha}$ is a Moore space, there exists a positive integer $q$ such that if $g \in C^\mu_{\alpha}$ and $x^\mu_{i\alpha} \in g$, then $g \subseteq \mathfrak{W}^\gamma_{\gamma q}$; clearly $q$ may be taken so that $q > m$. Now suppose that $h \in \mathcal{C}_q$ and $C^\mu_{\alpha} \subseteq h$. As $q > m$, for some $\gamma$, $h = \mathfrak{W}^\mu_{\gamma q}$; since $x^\mu_{i\alpha} \in \mathfrak{W}^\mu_{\gamma q}$, then $g^\mu_{\gamma q} \subseteq \mathfrak{W}^\mu_{\beta n}$ and therefore $\mathfrak{W}^\gamma_{\gamma q} \subseteq \mathfrak{W}^\mu_{\beta n}$. Thus if $h \in \mathcal{C}_n$ and $C^\mu_{\alpha} \subseteq h$, then $h \subseteq \mathfrak{U}$.

The space $X$ is a Moore space; that every real-valued continuous function on $X$ is constant may be proved exactly as in Hewitt [1].

REFERENCES


STATE UNIVERSITY OF IOWA