THE COEFFICIENT PROBLEM OF REGULAR FUNCTIONS

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1. Introduction. Let \( f(z) \) be a function meromorphic on \( 0 \leq \rho \leq |z| \leq 1 \). Let \( Rf(z) \) be non-negative or change sign a finite number of times on \( |z| = 1 \) except for poles. Then the coefficients of the Laurent expansion of \( f(z) \) in \( \rho' < |z| < 1 \) have been studied in a previous paper [3, Theorem 2].

In this note we shall study the coefficients under weaker conditions than those of the paper cited above.

2. Preliminaries. Let \( \sigma_k, k = 1, \ldots, 2s, \) denote \( 2s \) real numbers such that \( |\sigma_k| \leq \pi, k = 1, \ldots, 2s \) and set

\[
g(s, z) = Ks^{-\pi} \prod_{k=1}^{2s} (e^{i\sigma_k} - z)
\]

where \( K \) is a constant of modulus unity. Then we may give two definitions.

**Definition 1.** A function \( f(z) \) regular in \( 0 \leq \rho \leq |z| < 1 \) is said to belong to a class \( A(S, M) \), if

\[
\int_{-\pi}^{+\pi} |R(\rho e^{i\phi})| \, d\phi < c < \infty
\]

for \( \rho < r < 1 \), where \( T(z) = f(z)g(s, z) \) and if, for a non-negative constant \( M \),

\[
\lim_{r \to 1} \int_{-\pi}^{+\pi} [R(\rho e^{i\phi})]^- \, d\phi \leq 2\pi M
\]

where \( [R(\rho)]^- = \text{Max} \{0, -R(\rho)\} \).

**Definition 2.** A function \( f(z) \) regular in \( 0 \leq \rho \leq |z| < 1 \) is said to belong to a class \( B'(S, M) \) if \( g(s, z) \) satisfies the following: (a) both 0 and \( \pi \) are members of the set \( \sigma_k, k = 1, \ldots, 2s \); (b) if \( e^{i\sigma_k} \) for \( k = 1, 2, \ldots, 2s \) \((\sigma_k \neq 0, \pi)\) is an \( l \)th order zero, then \( e^{-i\sigma_k} \) is also an \( l \)th order zero, \( l \) being a positive integer, and if \( f(z) \) is a member of the class \( B(S, M) \).

**Lemma.** Let \( u(z) \) be harmonic in \( |z| < 1 \). Suppose that \( \lim \inf_{x \to z'} u(z) \geq 0 \) for every boundary point \( z' \) except for a finite number of boundary

Received by the editors December 8, 1959 and, in revised form, February 19, 1960.
Let $u(z) = O\left(\left| z - z_k \right|^{-1}\right)$ as $z \to z_k$, $k = 1, \ldots, m$. Then

$$\lim_{r \to 1} \int_{-\pi}^{+\pi} \left| u(re^{i\phi}) \right| d\phi < k < \infty.$$  

This lemma has been proved in [3, Lemma 4].

3. Theorems. If we use notations due to Goodman and Robertson

$$\Delta(s, k, n) = 2 \prod_{v(wk) = 0} (n^2 - v^2)/(s + k)! (s - k)! \quad (s \geq k > 0),$$  

$$\Delta(s, 0, n) = \prod_{v=1}^{s} (n^2 - v^2)/(s!)^2,$$

$$D(s, k, n) = 2k(n + s)!/(s + k)! (s - k)! (n - s - 1)! (n^2 - k^2) \quad (s \geq k > 0),$$

we obtain the following main Theorems 1 and 2.

**Theorem 1.** Let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

be regular in $0 \leq \rho \leq \left| z \right| < 1$ and be a member of the class $B(S, M)$. Then for all $n > s$, the following inequalities hold

$$|a_n + \delta a_{-n}| \leq \sum_{k=0}^{s-1} \Delta(s, k, n) |a_k + \delta a_{-k}|$$

$$+ \Delta(s, s, n)(|a_s + \delta (a_{-s} + K^2 M)| + M)$$

where $\delta = \frac{K^2 e^{-i \sum_{k=1}^{2s} \sigma_k}}{\sum_{k=1}^{2s} \sigma_k}$. In case $M = 0$ and $\sigma_k = 0$, $k = 1, 2, \ldots, 2s$, these equality signs hold.

A theorem of Robertson [5, Theorem 1] in which the imaginary part of $f(z)$ changes the sign $2P$ times is obtained by putting $M = 0$ and $\delta = 1$.

**Proof.** Set

$$e^{\alpha/2} = K \exp\left(\frac{i}{2} \sum_{k=1}^{2s} \sigma_k\right), \quad f_1(z) = \sum_{n=-\infty}^{\infty} e^{\alpha/2} a_n z^n = \sum_{n=-\infty}^{\infty} a'_n z^n$$

and

$$G(z) = f_1(z) - \sum_{n=1}^{\infty} (a'_{-n} z^{-n} - \bar{a}'_{-n} z^n).$$
Then $G(z)$ is regular in $|z| < 1$. The series of the right side of (3.5) is regular and its real part is identically zero on $|z| = 1$. If we put

$$F(z) = e^{-i/2}g(s, z)G(z) = \sum_{n=-\infty}^{\infty} A_n z^n$$

and

$$E(z) = F(z) - \sum_{n=1}^{\infty} (A_{-n} z^{-n} - A_{-n} z^n),$$

then $E(z)$ is regular in $|z| < 1$ and for $\rho < r < 1$

$$\int_{-\pi}^{\pi} |RE(re^{i\phi})| \, d\phi = \int_{-\pi}^{\pi} |RF(re^{i\phi})| \, d\phi$$

$$= \int_{-\pi}^{\pi} \left| RT(re^{i\phi}) - R\left\{ \sum_{n=1}^{\infty} (a'_{-n}(re^{i\phi})^{-n} - a'_{-n}(re^{i\phi})) \right\} \right. \exp \left( -\frac{i}{2} \sum_{k=1}^{2k} \sigma_k \right) g(s, re^{i\phi}) \right| \, d\phi.$$

Therefore, from the hypothesis the integral of the left side of (3.8) is bounded for $\rho < r < 1$. Hence $RE(z)$ is represented as the difference of two non-negative harmonic functions in $|z| < 1$. From Poisson's representation theorem for the non-negative harmonic function, we get the relation of (2.4) for $E(z)$. By F. Nevanlinna's theorem [4], the limit $\alpha(\theta) = \lim_{r \to 1} \int_{0}^{\pi} RE(re^{i\phi}) \, d\phi$, $|\theta| \leq \pi$, exists except for at most a countable set of $\theta$. The function $\alpha(\theta)$ is of bounded variation and $E(z)$ is represented as a Stieltjes’ integral $(1/2\pi)\int_{-\pi}^{\pi} (e^{i\theta} + z)/(e^{i\theta} - z) \alpha(\theta) \, d\theta + iE(0)$. Obviously $\alpha(\theta)$, $|\theta| \leq \pi$, is a difference of two monotone increasing functions $\alpha_1(\theta)$ and $\alpha_2(\theta)$, that is, $\alpha(\theta) = \alpha_1(\theta) - \alpha_2(\theta)$ where $\alpha(-\pi) = \alpha_1(-\pi) = \alpha_2(-\pi) = 0$ and the total variation of $\alpha$ in $(-\pi, \theta)$ is equal to $\alpha_1 + \alpha_2$. Moreover

$$\alpha_2(\pi) = \lim_{r \to 1} \frac{1}{2} \int_{-\pi}^{\pi} \left\{ \left| RE(re^{i\phi}) \right| - \left| RE(re^{i\phi}) \right| \right\} \, d\phi$$

$$= \lim_{r \to 1} \int_{-\pi}^{\pi} \left[ RE(re^{i\phi}) \right]^{-d\phi}$$

$$= \lim_{r \to 1} \int_{-\pi}^{\pi} \left[ RT(re^{i\phi}) \right] d\phi \leq 2\pi M.$$
\[ L(z) = E(z) + \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\alpha_2(\theta) \]

then \( R(j) \geq 0 \) for \( |z| < 1 \). By using the results of Robertson \[5\] and the author \[3\], we find that for \( n > s \)

\[ G(z) + \frac{e^{\alpha/2}}{2\pi} \int_{-\pi}^{+\pi} \frac{e^{i\theta} + z}{(e^{i\theta} - z)g(s, z)} \, d\alpha_2(\theta) = \sum_{n=0}^{\infty} B_n z^n \]

(3.11) \( \ll \sum_{n=s+1}^{\infty} \left( \sum_{k=0}^{r-1} \Delta(s, k, n) \left| a_k + e^{-\alpha}\tilde{a}_k \right| 
+ \Delta(s, s, n) \left| a_s + e^{-\alpha}(\tilde{a}_s + K^2M) \right| \right) z^n. \)

Moreover, we get

(3.12) \( \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{e^{i\theta} + z}{(e^{i\theta} - z)g(s, z)} \, d\alpha_2(\theta) \ll \sum_{n=s+1}^{\infty} \Delta(s, s, n) M z^n. \)

Thus from (3.11) and (3.12) we get (3.4). The majorant series of (3.11) is attained if \( \sigma_k = 0, k = 1, \ldots, 2s \).

**Corollary 1.** Let \( f(z) \) of the form (3.3) be regular in \( 0 \leq \rho \leq |z| < 1 \). Suppose that \( \lim \inf R(jf(z)) \geq 0 \) for every boundary point except for a finite number of points \( z_1, \ldots, z_m \) on \( |z| = 1 \) where \( |\gamma| = 1 \). Let \( \tau_k, k = 1, \ldots, m, \) be non-negative numbers, \( R(jf(z)) = O \left( |z - z_k|^{-\tau_k} \right) \), \( k = 1, \ldots, m \) and \( l_k = [\tau_k/2], k = 1, \ldots, m \). Finally let for \( t \) points \( z_{m,v} = e^{i\tau_{m,v}}, v = 1, \ldots, t, \) among \( z_k, k = 1, \ldots, m, \)

\[ \lim_{r \to 1} (-1)^{l_{m,s}} \prod_{k(m,s,v)=1}^{m} \left| z_k - z_{m,v} \right|^{2l_k(1-r)2l_{m,v}+1} Rf(re^{i\tau_{m,v}}) = \mu_{m,v} < 0 \]

and let the above limits be non-negative for the other \( m-t \) points among \( z_k, k = 1, \ldots, m. \) Then for \( n > s = 2 \sum_{k=1}^{m} l_k \)

\[ |a_n + \tilde{a}_{-n}| \leq \sum_{k=0}^{t} \Delta(s, k, n) |a_k + \tilde{a}_{-k}| 
- 2\Delta(s, s, n) \sum_{v=1}^{t} \mu_{m,v}. \]

(3.13)

If \( t = 0 \), these bounds are sharp.

**Proof.** By the hypothesis, we may take

(3.14) \[ g(s, z) = \prod_{k=1}^{m} \left( (z_k - z) \left( \frac{1}{z_k} - \frac{1}{z} \right) \right)^{l_k}. \]
Therefore, putting \( F(z)g(s, z)G(z) = \sum_{n=-s}^{\infty} A_n z^n \) for \( G(z) \) of (3.5) with \( e^{i\theta} = \gamma \), \( F(z) \) satisfies the hypothesis of the lemma. Hence using a theorem by H. Herzig [2] for \( E(z) \) of the form (3.7), we find that for \( z_{m_v}, v = 1, \ldots, t \) and \( \alpha_2(\theta) \) of \( \S 3 \),

\[
\lim_{r \to 1} (1 - r) R \mathcal{E}(re^{i \theta m_v}) = \frac{1}{2\pi} \left( \alpha_2(\theta_{m_v} + 0) - \alpha_2(\theta_{m_v} - 0) \right) = \mu_{m_v} < 0
\]

and for the other \( m-t \) points among \( z_k, k = 1, \ldots, m \), the above limits are non-negative. Therefore from Theorem 1 we get (3.13). And the bounds of (3.13) are sharp if \( t = 0 \).

**Theorem 2.** Let

\[
f(z) = \sum_{n=1}^{\infty} a_n z^n
\]

be regular in \( |z| < 1 \). Suppose that \( f(z) \) is a member of the class \( \mathcal{B}'(S, M) \) and that \( \kappa a_k, k = 1, \ldots, s-1 \), are imaginary. Then for \( n > s \)

\[
|a_n| \leq \sum_{k=1}^{s} D(s, k, n) |a_k| + 2D(s, s, n)M.
\]

If \( M = 0 \), these bounds are sharp.

A theorem of Goodman and Robertson [1] on the class \( \mathcal{T}(P) \) is obtained by putting \( M = 0 \).

**Proof.** Let \( g(s, z) \) be a function defined in Definition 2. Set

\[
F(z) = g(s, z) \sum_{n=1}^{\infty} a_n z^n = \sum_{n=-s+1}^{\infty} A_n z^n.
\]

Moreover, set

\[
E(z) = F(z) - \sum_{n=1}^{s-1} (A_{-n} z^{-n} - \overline{A}_{-n} z^n).
\]

Then by the argument used in the proof of Theorem 1, we have \( E(z) \in \mathcal{B}'(0, M) \). Using \( \alpha_2(\theta) \) in the proof of Theorem 1, we can write

\[
L(z) = E(z) + \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\alpha_2(\theta).
\]

Obviously \( RL(z) \geq 0 \) for \( |z| < 1 \). Hence from the hypothesis that \( \kappa a_k \),
\( k=1, \cdots, s-1 \), are imaginary and the hypothesis for \( g(s, z) \), follows that for \( n>s \)

\[
f(z) + \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{e^{i\theta} + z}{(e^{i\theta} - z)g(s, z)} \, d\alpha_2(\theta) = \sum_{n=1}^{\infty} B_n z^n
\]

(3.20)

\[
\ll \sum_{n=s+1}^{\infty} \left( \sum_{k=1}^{s} D(s, k, n) |a_k| + D(s, s, n) |Ka_s - M| \right) z^n
\]

where \( M = (1/2\pi) \int_{-\pi}^{+\pi} d\alpha_2(\theta) \) if we apply the argument due to Goodman and Robertson [1] and the author [3]. Thus for \( n>s \)

\[
f(z) \ll \sum_{n=s+1}^{\infty} \left( \sum_{k=1}^{s} D(s, k, n) |a_k| + 2D(s, s, n)M \right) z^n.
\]

Hence we get (3.17). The majorant series of (3.20) is sharp. Therefore, if \( M=0 \), the bounds of (3.17) are sharp.

**Corollary 2.** Let \( f(z) \) of the form (3.16) be regular in \(|z|<1\) and let \( f(z)g(s, z) \) satisfy the conditions of the lemma with \( g(s, z) \) as in Definition 2. Suppose that for \( t \) points \( z_k = e^{i\theta_k}, \ k=1, \cdots, t, \) \( \lim_{r \to 1} (1-r)g(s, re^{i\theta_k})f(re^{i\theta_k}) = \mu_k < 0 \) and for the other \( m-t \) points among \( z_k, k=1, \cdots, m, \) the above limits are non-negative and that \( Ka_k, k=1, \cdots, s-1, \) are imaginary. Then for \( n>s \)

\[
|a_n| \leq \sum_{k=1}^{s} D(s, k, n) |a_k| - 2D(s, s, n) \sum_{\nu=1}^{t} \mu_{\nu}.
\]

If \( t=0 \), these bounds are sharp.

The proof of this corollary is obtained by the same method as in the proof of Corollary 1.

Two theorems of Robertson and Goodman [1, Theorem 4] and Robertson [5, Theorem 2] are generalized by the following Theorems 4 and 3 respectively.

**Theorem 3.** Let \( f(z) \) of the form (3.3) be regular in \( 0 \leq \rho \leq |z| <1 \) and let \( z(df(z)/dz) \in B'(S, M) \). Then for \( n>s \)

(3.21) \[
|a_n + \delta a_{-n}| \leq \sum_{k=1}^{s} D(s, k, n) |a_k + \delta a_{-k}| + (2/n)\Delta(s, s, n)M,
\]

where \( \delta = K^2 \exp \left[ -i(\sum_{k=1}^{2s} \sigma_k - \pi) \right]. \) If \( M=0 \) and \( \sigma_k = 0, \ k=1, \cdots, 2s, \) these equality signs hold.

**Theorem 4.** Let \( f(z) \) of the form (3.3) be regular in \( 0 \leq \rho \leq |z| <1 \) and let \( zdf(z)/dz \in B'(S, M) \). Suppose that \( Ka_k + \bar{K}a_{-k}, \ k=1, \cdots, s-1, \) are imaginary. Then for \( n>s \)
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\[ |a_n + \bar{K}_n a_{-n}| \]

(3.22)
\[ \leq \sum_{k=1}^{n} \frac{k}{n} D(s, k, n) |a_k + \bar{K}_n a_{-k}| + \frac{2}{n} D(s, s, n) M. \]

If \( M = 0 \), then these bounds are sharp.

The proofs of Theorems 3 and 4 are easily obtained.

**Theorem 5.** Let

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]

and \( h(z) = \sum_{n=0}^{\infty} d_n z^n, d_0 \neq 0 \), be regular in \( |z| < 1 \), and \( h(z) \in B(S, M) \).

Let \( f(z)/h(z) \) be regular in \( |z| < 1 \) and be a member of the class \( B(0, M') \).

Then for \( n > s \)

\[ |a_n| \leq \left\{ 2 \sum_{l=0}^{n-1} \left( \sum_{k=0}^{s} \Delta(s, k, l) |d_k| + 2\Delta(s, s, l) M \right) + 2 \sum_{k=0}^{s-1} |d_k| + \sum_{k=0}^{s} \Delta(s, k, n) |d_k| + 2\Delta(s, s, n) M \right\} \left( 2M' + \left| \frac{a_0}{d_0} \right| \right). \]  

**Proof.** From the hypothesis, we find

\[ \frac{f(z)}{h(z)} = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\alpha(\theta) + i \frac{a_0}{d_0} \]

where \( \alpha(\theta) \) is a function of bounded variation of \( \theta \). Let \( \alpha_2(\theta) \) be the function of §3. Then \( (1/2\pi i) \int_{-\pi}^{\pi} d\alpha_2(\theta) = M' \). And

\[ \frac{f(z)}{h(z)} \leq \left( 2M' + \left| \frac{a_0}{d_0} \right| \right) \frac{1 + z}{1 - z}. \]  

Thus from (3.4) and Theorem 1, we get (3.24).

**Theorem 6.** Let \( f(z) \) of the form (3.16) be regular in \( |z| < 1 \). Suppose that \( h(z) = \sum_{n=1}^{\infty} d_n z^n, d_1 \neq 0 \), satisfies the hypothesis of Theorem 2. Let \( f(z)/h(z) \) be regular in \( |z| < 1 \) and be of the class \( B(0, M') \). Then for \( n > s \)

\[ |a_n| = \left\{ 2 \sum_{l=1}^{n-1} \left( \sum_{k=1}^{s} D(s, k, l) |d_k| + 2D(s, s, l) M \right) + 2 \sum_{k=1}^{s-1} d_k + \sum_{k=1}^{s} D(s, k, n) |d_k| + 2D(s, s, n) M \right\} \left( 2M' + \left| \frac{a_1}{d_1} \right| \right). \]

The proof of this theorem is obtained by using Theorem 2.
4. Remark. Modifying \( g(s, z) \) slightly in §1, we get more precise versions of Theorems 1 and 2. Let \( \sigma_k, k = 1, \cdots, 2s \), be real numbers such that \( |\sigma_k| \leq \pi, k = 1, \cdots, 2s \), and set for \( |K| = 1 \) and \( p \geq s \),

\[
(4.1) \quad g(s, p, z) = Kz^{s-p} \prod_{k=1}^{p} \left( e^{i\sigma_k} - z \right) / \prod_{k=p+1}^{2s} \left( e^{i\sigma_k} - z \right).
\]

Then we can define the following two classes.

**Definition 3.** A function \( f(z) \) regular in \( 0 \leq \rho \leq |z| < 1 \) is said to belong to a class \( B(S, P, M) \) if the relation (2.2) and (2.3) hold when \( g(s, z) \) is replaced by \( g(s, p, z) \).

**Definition 4.** A function \( f(z) \) regular in \( 0 \leq \rho \leq |z| < 1 \) is said to belong to a class \( B'(S, P, M) \) if \( g(s, p, z) \) satisfies the following:

(a') both 0 and \( \pi \) are members of the set \( \sigma_k, k = 1, \cdots, 2s \); (b') if \( e^{i\sigma_k}(\sigma_k \neq 0, \pi) \) for each \( k \) is an \( l \)th order zero or pole, \( e^{-i\sigma_k} \) is also an \( l \)th order zero or pole respectively, \( l \) being a positive integer, and if \( f(z) \) is a member of a class \( B'(S, P, M) \).

Let \( f(z) \) of the form (3.3) be regular in \( 0 \leq \rho \leq |z| < 1 \) and let \( f(z) \) be a member of the class \( B(S, P, M) \). Then by the argument used in the proof of Theorem 1, the upper bounds of the moduli of \( a_n + \delta a_n \) for all \( n > s \), are obtained in terms of \( M \) and \( |a_k + \delta a_{-k}|, k = 0, 1, \cdots, p-s \). Similarly, if \( f(z) \) of the form (3.16) is regular in \( |z| < 1 \) and is a member of the class \( B'(S, P, M) \) and if \( K\sigma_k, k = 1, \cdots, p-s-1 \) are imaginary, then, by using the argument in the proof of Theorem 2, we get the upper bounds of the moduli of \( a_n \) for all \( n > s \), in terms of \( M \) and \( a_k, k = 1, \cdots, p-s \) (see the author [3]).

**References**


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