SOLUTIONS OF LINEAR DIFFERENTIAL SYSTEMS SATISFYING BOUNDARY CONDITIONS IN THE LARGE

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1. Introduction. In recent years existence and uniqueness theorems have been given for differential systems where multiple-point boundary conditions are imposed. For those theorems which apply to the linear system

\[ y_i' = \sum_{j=1}^{n} a_{ij}(x)y_j + b_i(x), \quad i = 1, \ldots, n, \]

the interval over which the boundary points are distributed is restricted in length. In the present paper conditions on the \( a_{ij}(x) \) are given which assure a unique solution satisfying two-point and three-point boundary conditions where these points are required only to belong to the interval, say \([a, b]\), over which the \( a_{ij}(x), b_i(x) \) are continuous.

2. Two-point boundary conditions. For points \( \alpha_1, \alpha_2, \alpha_1 < \alpha_2, \) of \([a, b]\) we define the following conditions over \([\alpha_1, \alpha_2]\):

A. \( a_{ij}(x), i \neq j, \) is nonzero.

B. If \( a_{mn}(x) > 0, a_{mk}(x) \) has the same sign as \( a_{kn}(x); \) if \( a_{mn}(x) < 0, a_{mk}(x) \) has the opposite sign to \( a_{kn}(x); \) \( a_{nk}(x) \) has the same sign as \( a_{kn}(x), m, k = 1, \ldots, n - 1, m \neq k. \)

**Theorem 1.** Let the \( a_{ij}(x) \) be continuous and satisfy (A), (B) for some \( \alpha_1, \alpha_2 \) of \([a, b]\). Then there exists a unique solution of (1) satisfying the conditions

\[ y_k(\alpha_1) = \beta_k, \quad y_n(\alpha_2) = \beta_n, \quad k = 1, \ldots, n - 1, \]

where \( \beta_1, \ldots, \beta_n \) are arbitrary real numbers.

**Proof.** Let \((y_{i1}(x), \ldots, y_{in}(x)), i = 1, \ldots, n, \) be solutions of the homogeneous system

\[ y_i' = \sum_{j=1}^{n} a_{ij}(x)y_j \]

satisfying

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1 A dual theorem may be given for \( \alpha_2 < \alpha_1 \) if (B) is altered so that, if \( a_{mn}(x) > 0, a_{mk}(x) \) is required to have the opposite sign to \( a_{kn}(x) \) and, if \( a_{mn}(x) < 0, a_{mk}(x) \) is required to have the same sign as \( a_{kn}(x). \)
\begin{equation}
y_{ij}(\alpha_i) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad i, j = 1, \ldots, n.
\end{equation}

Then the general solution \((y_1(x), \ldots, y_n(x))\) of (1) is given by

\[ y_i(x) = c_1y_{1i}(x) + c_2y_{2i}(x) + \cdots + c_ny_{ni}(x) + y_{pi}(x), \]

where \((y_{pi}(x), \ldots, y_{pn}(x))\) is a particular solution of (1). Imposing the boundary conditions (2) to this system and simplifying according to (4), we obtain

\[ c_1 = \beta_1 - y_{pi}(\alpha_1) \]

\[ \vdots \]

\[ c_{n-1} = \beta_{n-1} - y_{pn-1}(\alpha_1) \]

\[ c_1y_{1n}(\alpha_2) + c_2y_{2n}(\alpha_2) + \cdots + c_ny_{nn}(\alpha_2) = \beta_n - y_{pn}(\alpha_2). \]

This system has a solution, hence (2) can be satisfied uniquely, if \(y_{nn}(\alpha_2) \neq 0\).

Assume at least one of \(y_{n1}(x), \ldots, y_{nn}(x)\) has a zero on \((\alpha_1, \alpha_2)\). Of the \(a_{mn}(x), m = 1, \ldots, n-1,\) let \(a_{p_0n}(x), p = 1, \ldots, r,\) be those, if any, which are positive and let \(a_{p_1n}(x), q = 1, \ldots, s,\) be those, if any, which are negative. Then, by (4), \(y_{nn}(\alpha_1) = 1, y'_{nq}(\alpha_1) > 0, y''_{nq}(\alpha_1) < 0.\) Hence \(y_{nn}(x), y'_{nq}(x) > 0, y''_{nq}(x) < 0\) to the immediate right of \(x = \alpha_1.\) Now since the \(y_{nj}(x)\) are continuous it is possible to let \(c\) be the smallest zero of any of these functions on \((\alpha_1, \alpha_2).\) If \(y_{ne}(c) = 0, n = 0\) or \(1 \leq e \leq r,\) then \(y'_{ne}(c) = 0.\) But, under this assumption,

\[ y_{ne}(c) = a_{p_01}(c)y_{n1}(c) + \cdots + a_{p_{r-1}}(c)y_{nr_{r-1}}(c) \]

\[ + a_{p_{r+1}}(c)y_{nr_{r+1}}(c) + \cdots + a_{p_s}(c)y_{ns}(c) \]

\[ + a_{q1}(c)y_{nq1}(c) + \cdots + a_{q_s}(c)y_{nqs}(c) \]

\[ + a_{en}(c)y_{nn}(c) \]

is positive since, by the hypotheses and the above determined properties of \(y_{nj}(x),\)

\[ a_{p_0}(c)y_{np_0}(c), a_{q1}(c)y_{nq1}(c), a_{en}(c)y_{nn}(c) \geq 0, \]

and since \(y_{n1}(x), \ldots, y_{nr_{r-1}}(x), y_{nr_{r+1}}(x), \ldots, y_{nn}(x)\) cannot all vanish at \(x = c.\) If \(y_{num}(c) = 0, 1 \leq f \leq s,\) then \(y'_{num}(c) \geq 0.\) But, under this assumption, \(y'_{num}(c) < 0\) since

\[ a_{p_0}(c)y_{np_0}(c), a_{p_f}(c)y_{np_f}(c), a_{p_m}(c)y_{nn}(c) \leq 0 \]

and the functions \(y_{n1}(x), \ldots, y_{nr_{r-1}}(x), y_{nr_{r+1}}(x), \ldots, y_{nn}(x)\) cannot all vanish at \(x = c.\) We now have a contradiction on the choice of \(c.\)
Hence none of \( y_{n1}(x), \ldots, y_{nn}(x) \) vanishes on \((\alpha_1, \alpha_2]\) and the theorem follows.

**Corollary.** Let the \( a_{ij}(x) \) satisfy (A), (B) over \([a, b]\). Then Theorem 1 is valid without restricting the boundary points \( \alpha_1, \alpha_2 \) further than requiring them to belong to \([a, b]\).

Conditions are not imposed on \( a_{ij}(x), i = 1, \ldots, n \). If \( a_{\gamma\gamma_i}(x) \equiv 0 \) over \((\alpha_1, \alpha_2)\) for \( i = 1, \ldots, n \), included in (B), we need require the positive functions only to be nonnegative and the negative functions to be only nonpositive with \( a_{\gamma\gamma_i}(\alpha_2) \neq 0 \).

3. **Three-point boundary conditions.** For points 

\[ \alpha_1, \alpha_2, \alpha_3 (\alpha_1 \leq \alpha_2 \leq \alpha_3) \]

of \([a, b]\) we define the following conditions:

C. \( a_{mm}(x) = 0 \) on \((\alpha_1, \alpha_3)\); \( a_{1n}(x), a_{n1}(x) > 0 \) on \([\alpha_1, \alpha_3]\)

\[ m = 2, \ldots, n - 1. \]

D. For each \( m, 2 \leq m \leq n - 1 \), either

\[
\begin{align*}
\text{(1)} & \quad a_{m1}(x) \geq 0 \quad \text{on} \quad (\alpha_1, \alpha_3), \\
& \quad a_{m1}(\alpha_2) > 0 \quad \text{or} \\
& \quad a_{m1}(x) \begin{cases} 
\leq 0 & x \in (\alpha_1, \alpha_2], \\
\geq 0 & x \in [\alpha_2, \alpha_3).
\end{cases}
\end{align*}
\]

If (1) holds then

\[
\begin{align*}
& \quad a_{1m}(x) > 0 \quad \text{on} \quad [\alpha_1, \alpha_2], \\
& \quad a_{nm}(x) \begin{cases} 
< 0 & x \in [\alpha_1, \alpha_2), \\
= 0 & x = \alpha_2, \\
> 0 & x \in (\alpha_2, \alpha_3],
\end{cases} \\
& \quad a_{mn}(x) \begin{cases} 
\leq 0 & x \in (\alpha_1, \alpha_2], \\
\geq 0 & x \in [\alpha_2, \alpha_3).
\end{cases}
\end{align*}
\]

If (2) holds then

\[
\begin{align*}
& \quad a_{1m}(x) \begin{cases} 
< 0 & x \in [\alpha_1, \alpha_2), \\
= 0 & x = \alpha_2, \\
> 0 & x \in (\alpha_2, \alpha_3],
\end{cases} \\
& \quad a_{nm}(x) \begin{cases} 
\leq 0 & x \in (\alpha_1, \alpha_2], \\
\geq 0 & x \in [\alpha_2, \alpha_3).
\end{cases}
\end{align*}
\]
E. For the case D(1):

\[
\begin{align*}
\alpha_m(x) &< 0 & \text{on } (\alpha_1, \alpha_2), \\
\alpha_m(x) &> 0 & \text{on } [\alpha_2, \alpha_3],
\end{align*}
\]

For the case D(2):

\[
\begin{align*}
\alpha_m(x) &< 0 & \text{on } (\alpha_1, \alpha_2), & \text{if } \alpha_1(\alpha_2) > 0, \\
\alpha_m(x) &> 0 & \text{on } [\alpha_2, \alpha_3], & \text{if } \alpha_1(\alpha_2) = 0.
\end{align*}
\]

F. For each \( m, 2 \leq m \leq n - 1 \), there exists a neighborhood, \((\alpha_2 - \delta_m, \alpha_2 + \delta_m)\), of \( \alpha_2 \) for which \( a_m(x), \ldots, a_{mn}(x) \) do not all have a common zero.

**Lemma 1.** Let the \( a_{ij}(x) \) be continuous and satisfy (C), (D), (E), (F) for some \( \alpha_1, \alpha_2, \alpha_3 \) of \([a, b]\) and let \( (y_{11}(x), \ldots, y_{1n}(x) \) be the solution of (3) satisfying

\[
y_{1j}(\alpha_2) = \delta_{1j}, \quad j = 1, \ldots, n.
\]

Then \( x = \alpha_2 \) is an isolated zero of \( y_{1j}(x) \), \( f = 2, \ldots, n \).

**Proof.** Of the \( a_{ij}(x) \), \( f = 2, \ldots, n \), let \( a_{ij}(x) \), \( \nu_1 < \nu_2 < \cdots < \nu_r \) be those which are positive at \( \alpha_2 \) and let \( a_{ij}(x) \), \( \mu_1 < \mu_2 < \cdots < \mu_s \), be those, if any, which are zero at \( \alpha_2 \). Then, by (5), \( y_{11}(\alpha_2) = 1, y'_{1p}(\alpha_2) > 0, p = 1, \cdots, r \). Since \( y_{1r}(x) \) is continuous and vanishes at \( x = \alpha_2 \), there exists a \( \delta > 0 \) such that

\[
\begin{align*}
y_{11}(x), y_{1r}(x) &< 0 & \text{on } (\alpha_2 - \delta, \alpha_2), \\
y_{11}(x), y_{1r}(x) &> 0 & \text{on } (\alpha_2, \alpha_2 + \delta).
\end{align*}
\]

With this and the hypotheses, we have that

\[
y'_{1\mu_1}(x) = a_{\mu_11}(x)y_{11}(x) + a_{\mu_1\nu_1}(x)y_{1\nu_1}(x) + \cdots + a_{\mu_1r}(x)y_{1r}(x)
\]

is negative over \((\alpha_2 - \delta_1, \alpha_2)\) and positive over \((\alpha_2, \alpha_2 + \delta_1)\), where \( \delta_1 = \min(\delta, \delta_{\mu_1}) \). Hence, since \( y_{1\mu_1}(x) \) vanishes at \( x = \alpha_2 \) and is continuous, \( y_{1\mu_1}(x) > 0 \) on \((\alpha_2 - \delta_1, \alpha_2)\), \((\alpha_2, \alpha_2 + \delta_1)\). We now have that
\[ y'_{my}(x) = a_{my}(x)y_{11}(x) + a_{my_1}(x)y_{12}(x) + a_{my_2}(x)y_{13}(x) \\
+ \cdots + a_{my_r}(x)y_{1r}(x) \]

is negative over \((a_2 - \delta_2, a_2)\) and positive over \((a_2, a_2 + \delta_2)\), where \(\delta_2 = \min(\delta_1, \delta_2)\). This implies that \(y_{1m}(x) > 0\) on \((a_2 - \delta_2, a_2)\), \((a_2, a_2 + \delta_2)\). By continuing in this way we find \(y_{1m}(x) > 0\) in a neighborhood of \(x = a_2\) for \(q = 1, \ldots, s\).

We have shown that each \(y_{1f}(x)\) is either positive or negative to the immediate left of \(x = a_2\) and positive to the immediate right of \(x = a_2\). Hence \(a_2\) is an isolated zero of \(y_{1f}(x), f = 2, \ldots, n\).

**Lemma 2.** Let the \(a_{ij}(x)\) be continuous and satisfy (C), (D), (E), (F) for some \(\alpha_1, \alpha_2, \alpha_3\) of \([a, b]\) and let \((y_{n1}(x), \ldots, y_{nn}(x))\) be the solution of (3) satisfying

\[ y_{nj}(a_2) = \delta_{nj}, \quad j = 1, \ldots, n. \]

Then \(x = a_2\) is an isolated zero of \(y_{ne}(x), e = 1, \ldots, n-1\).

**Proof.** By a process similar to that in the proof of Lemma 1, we find that \(y_{nr_p}(x), p = 1, \ldots, r,\) is positive in a neighborhood of \(x = a_2\) and \(y_{nr_q}(x), q = 1, \ldots, s,\) is negative to the immediate left of \(x = a_2\), positive to the immediate right of \(x = a_2\).

**Theorem 2.** Let the \(a_{ij}(x)\) be continuous and satisfy (C), (D), (E), (F) for some \(\alpha_1, \alpha_2, \alpha_3\) of \([a, b]\). Then there exists a unique solution of (1) satisfying

\[ y_1(\alpha_1) = \beta_1, \quad y_m(\alpha_2) = \beta_m, \quad y_n(\alpha_3) = \beta_n, \quad m = 2, \ldots, n - 1, \]

where \(\beta_1, \ldots, \beta_n\) are arbitrary real numbers.

**Proof.** Let \((y_{i1}(x), \ldots, y_{in}(x)), i = 1, \ldots, n,\) be solutions of (3) satisfying

\[ y_{ij}(a_2) = \delta_{ij}, \quad i, j = 1, \ldots, n. \]

Then the general solution \((y_1(x), \ldots, y_n(x))\) of (1) is given by

\[ y_i(x) = c_1y_{1i}(x) + c_2y_{2i}(x) + \cdots + c_ny_{ni}(x) + y_{pi}(x), \]

where \((y_{pi}(x), \ldots, y_{pn}(x))\) is a particular solution (1). Imposing the boundary conditions (7) and simplifying according to (8), we obtain

\[ c_1y_{11}(\alpha_1) + c_2y_{21}(\alpha_1) + \cdots + c_ny_{n1}(\alpha_1) = \beta_1 - y_{p1}(\alpha_1), \]

\[ c_2 = \beta_2 - y_{p2}(\alpha_2), \]

\[ \vdots \]

\[ c_{n-1} = \beta_{n-1} - y_{pn-1}(\alpha_2), \]

\[ c_1y_{1n}(\alpha_3) + c_2y_{2n}(\alpha_3) + \cdots + c_ny_{nn}(\alpha_3) = \beta_n - y_{pn}(\alpha_3). \]
This system has a solution, hence (7) can be satisfied uniquely, if
\( y_1(\alpha_1) y_{n_1}(\alpha_3) - y_1(\alpha_3) y_{n_1}(\alpha_1) \) is nonzero. We proceed to show this is the case.

Let \( a_{\mu_1}(x), a_{\mu_2}(x) \) be defined as above. We first prove
(a) \( y_{11}(x) > 0 \) on \([\alpha_1, \alpha_2]\).

Assume at least one of the functions \( y_{11}(x), \ldots, y_{n_1}(x) \) has a zero on \([\alpha_1, \alpha_2]\). By virtue of Lemma 1 and the continuity of \( y_{1j}(x) \), \( j=1, \ldots, n \), it is possible to let \( c \) be the largest zero of any of these functions on \([\alpha_1, \alpha_2]\). Hence the sign of \( y_{1j}(x) \) as found in Lemma 1 holds over the interval \((c, \alpha_2)\).

If \( y_{11}(c) = 0 \), then, since \( y_{11}(x) > 0 \) on \((c, \alpha_2)\), \( y_{11}(c) = 0 \). But, under this assumption,

\[
y'_{11}(c) = a_{1r_1}(c)y_{1r_1}(c) + \cdots + a_{1r_r}(c)y_{1r_r}(c) \\
+ a_{1s_1}(c)y_{1s_1}(c) + \cdots + a_{1s_s}(c)y_{1s_s}(c)
\]

is negative since \( a_{1r_p}(c)y_{1r_p}(c), a_{1s_q}(c)y_{1s_q}(c) \leq 0 \) and the functions \( y_{12}(x), \ldots, y_{n_1}(x) \) cannot all vanish at \( x = c \). Hence \( y_{11}(c) \neq 0 \). For a similar reason, \( y_{1n}(c) \neq 0 \). The function \( y_{1q}(x), q=1, \ldots, s \), does not vanish at \( x = c \) since, from the proof of Lemma 1, its derivative does not change sign over \((c, \alpha_2)\). For any \( e, 1 \leq e \leq r-1 \),

\[
y'_{1e}(x) = a_{1r_1}(x)y_{11}(x) + a_{1s_1}(x)y_{1s_1}(x) + \cdots \\
+ a_{1s_{e-1}}(x)y_{1s_{e-1}}(x) + a_{1s_e}(x)y_{1s_e}(x)
\]

is positive to the immediate left of \( x = \alpha_2 \) and nonnegative over \((c, \alpha_2)\). Thus since \( y_{1e}(\alpha_2) = 0 \), \( y_{1e}(x) \) cannot vanish at \( x = c \). We now have a contradiction on the choice of \( c \). Hence \( y_{11}(x), \ldots, y_{n_1}(x) \) do not vanish on \([\alpha_1, \alpha_2]\).

The following statements are proved in a similar manner to (a):
(b) \( y_{n_1}(x) < 0 \) on \([\alpha_1, \alpha_2]\).
(c) \( y_{1n}(x) > 0 \) on \((\alpha_2, \alpha_3]\).
(d) \( y_{nn}(x) > 0 \) on \((\alpha_2, \alpha_3]\).

We now have \( y_{11}(\alpha_1) y_{nn}(\alpha_3) - y_{1n}(\alpha_3) y_{n_1}(\alpha_1) > 0 \) for \( \alpha_1 < \alpha_3 < \alpha_2 \). If \( \alpha_1 = \alpha_2 < \alpha_3 \) (\( \alpha_1 < \alpha_2 = \alpha_3 \)) then the determinant in question is \( y_{nn}(\alpha_3) > 0 \) (\( y_{11}(\alpha_1) > 0 \)). Hence there exists a unique solution for the \( c_i \). This in turn gives a unique solution of (1) satisfying (7).

**Corollary.** Let the \( a_{ij}(x) \) satisfy (C), (D), (E), (F) over the interval \([a, \alpha_2), (\alpha_2, b] \) for some \( \alpha_2 \subseteq [a, b] \). Then Theorem 2 is valid without
restricting the boundary points \( \alpha_1, \alpha_3 \) further than requiring them to belong to \([a, \alpha_2), [\alpha_2, b]\), respectively.

Conditions are not imposed on \( a_{11}(x), a_{nn}(x) \). If these functions are identically zero over \((\alpha_1, \alpha_3)\) Theorem 2 follows for weaker restrictions than \((C), (D)\). For the \( a_{1f}(x), a_{ne}(x), f=2, \ldots, n, e=1, \ldots, n-1\), it is sufficient to require that the positive functions be nonnegative, the negative functions be nonpositive and \( a_{1n}(\alpha_2), a_{n1}(\alpha_2) > 0 \).

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A MOORE SPACE ON WHICH EVERY REAL-VALUED CONTINUOUS FUNCTION IS CONSTANT

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F. B. Jones [2] recently gave an example of a Moore space \( \Lambda_{\infty} \) in which there exists a point \( x \) such that \( \Lambda_{\infty} \) is not completely regular at \( x \). It is easy to modify the construction used by Jones so as to obtain a Moore space \( A \) in which there exist distinct points \( a \) and \( b \) such that for every real-valued continuous function \( f \) on \( A \), \( f(a) = f(b) \). Upon applying Urysohn's process of condensation of the singularities of the space \( A \) [4], in a manner similar to that used by Hewitt [1], there results a Moore space \( X \) on which every real-valued continuous function is constant.

Throughout this paper, \( J \) denotes the set of positive integers. A sequence is a function on \( J \), and if \( f \) is a sequence and \( n \in J \), then \( f_n \) denotes \( f(n) \).

By a Moore space is meant a topological space \( X \) whose topology has a basis consisting of sets termed regions, satisfying the following condition (axiom 1, that is, parts 1, 2, and 3 of axiom 1, of [3]):

There exists a sequence \( G \) such that (1) if \( n \in J, G_n \) is a collection of regions covering \( X \), (2) if \( n \in J, G_{n+1} \subseteq G_n \), and (3) if \( r \) is a region, \( x \in r \), and \( y \in r \), then there exists a positive integer \( n \) such that if \( g \in G_n \) and \( x \in g \), then \( \bar{g} \subseteq (r - \{x\}) \cup \{y\} \). The following characterization of a Moore space will be used in this paper: \( X \) is a Moore space if and only if \( X \) is a regular Hausdorff space for which there exists a sequence \( G \) of open coverings of \( X \) such that if \( U \) is an open set and

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