SUMMABILITY-PRESERVING FUNCTIONS

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The questions answered in this paper suggest themselves naturally. The first lemma is a consequence of a result of R. C. Buck.

**Lemma.** Let $f$ be a function of a complex variable and $A$ a Toeplitz matrix such that $\{f(x^n)\}$ is $A$-summable whenever $\{x^n\}$ converges. Then $f$ is continuous.

**Proof.** Every subsequence of $\{x^n\}$ converges, so that every subsequence of $\{f(x^n)\}$ is $A$-summable, by hypothesis. By [1], applied to complex sequences, $\{f(x^n)\}$ is actually convergent. (For the proof of [1] can be modified to apply to complex sequences. Or, see [2, Theorem 2], for a proof in a slightly more general context.) Let $\{x^n\}$ converge to $x$. The sequence $x_1, x, x_2, x, \cdots$ is also convergent, so that the sequence $f(x_1), f(x), f(x_2), f(x), \cdots$ converges. This shows that $\{f(x^n)\}$ converges to $f(x)$ whenever $\{x^n\}$ converges to $x$. Thus the continuity of $f$ is proved.

The converse of the lemma is a consequence of the definition of Toeplitz matrix. The lemma is of course also true for real-valued functions of a real variable. The same comment applies also after the following theorem.

**Theorem.** Let $A = (a_{in}), 1 \leq i, n < \infty$, be a Toeplitz matrix, and $f$ a function of a complex variable, such that if $\{x^n\}$ is a bounded $(C, 1)$-summable sequence, then $\{f(x^n)\}$ is $A$-summable. Then $f$ is linear. If in addition $f$ is not the constant function, then $A$ sums every bounded $(C, 1)$-summable sequence to its $(C, 1)$ sum.

**Proof.** Every convergent sequence is a bounded $(C, 1)$-summable sequence. Thus $f$ satisfies the hypotheses of the lemma. Hence $f$ is continuous. We will prove that $f((a+b)/2) = (f(a)+f(b))/2$ for all $a$ and $b$. It is well known that a continuous function with this property is linear.

Consider the sequence $a, b, a, b, \cdots$, which is a bounded $(C, 1)$-summable sequence. In fact, we may interpolate any number of terms $(a+b)/2$ between successive terms of this sequence, infinitely many if we wish, and still have a bounded $(C, 1)$-summable sequence. To prove this, let us examine the average of the first $n$ terms of such an interpolated sequence. These first $n$ terms consist of $rb$'s, $r$ or $r-1$ a's,
and \( n - 2r \) or \( n - 2r - 1((a + b)/2)'s, say. The average of these \( n \) terms is therefore equal to \( (1/n)(r(a + b) + (n - 2r)(a + b)/2) \) or \( (1/n)(r(a + b) + a + (n - 2r - 1)(a + b)/2) \), that is, to \( (a + b)/2 \) or \( ((a + b)/2)(1 - 1/n) + a/n \). As \( n \) approaches infinity, the Cesaro means of such an interpolated sequence therefore converge to \( (a + b)/2 \). This proves the assertion. By the hypothesis of this theorem, we conclude that any sequence \( f(a), f(b), f(a), f(b), \ldots \), with any number of \( f((a + b)/2)'s \) interpolated into it is \( A \)-summable. We shall use this fact shortly.

If \( f \) is the constant function, there is nothing more to prove. Throughout the remainder of the proof, \( f \) will be nonconstant. Let \( a, b \) be such that \( f(a) = p, f(b) = q \), with \( p \neq q \). The sequence \( p, q, p, q, \ldots \) is \( A \)-summable by hypothesis. Then the sequence \( p - q, 0, p - q, 0, \ldots \) is also \( A \)-summable. For this sequence is obtained from the preceding sequence by subtracting the sequence \( q, q, q, \ldots \). Upon dividing each term of \( p - q, 0, p - q, 0, \ldots \) by \( p - q \), we conclude that the sequence \( 1, 0, 1, 0, \ldots \) is \( A \)-summable. In other words, \( \lim_{i \to \infty} \sum_{n \text{ odd}} a_{in} \) exists; call it \( r \). Since for every Toeplitz matrix, \( \lim_{i \to \infty} \sum_{n=1}^\infty a_{in} = 1 \), we must have \( \lim_{i \to \infty} \sum_{n \text{ even}} a_{in} = 1 - r \).

Now consider any sequence \( f(a), f(b), f(a), f(b), \ldots \). We shall interpolate \( f((a + b)/2)'s \) into this sequence in such a way that we obtain a new sequence, which we know to be \( A \)-summable by the above argument, and yet which has a subsequence of its sequence of auxiliary means under \( A \) convergent to \( f((a + b)/2) \), and another subsequence of its sequence of auxiliary means under \( A \) convergent to \( rf(a) + (1 - r)f(b) \). But since the sequence of auxiliary means under \( A \) of the interpolated sequence converges, and since a convergent sequence has but one limit point, we must conclude that these two limit points coincide.

Let \( M \) be the maximum of the absolute values of the three numbers \( f(a), f(b), f((a + b)/2). \) Let \( N_1 \) be even and so large that \( \sum_{n>N_1} |a_{in}| < 1/2M \). Then for any sequence \( \{c_n\} \) composed only of terms chosen from \( f(a), f(b), f((a + b)/2) \), whose first \( N_1 \) terms are the first \( N_1 \) terms of the sequence \( f(a), f(b), f(a), f(b), \ldots \), we observe that \( \sum_{n=1}^{N_1} a_{in}c_n \) differs in absolute value from \( f(a) \sum_{n \text{ odd}} a_{in} + f(b) \sum_{n \text{ even}} a_{in} \) by less than \( 2M \cdot 1/2M = 1 \). Let \( i_1 = 1 \) and choose \( i_2 > i_1 \) and so large that \( \sum_{n=1}^{N_1} |a_{i_2,n}| < 1/2M \). We now start interpolating terms \( f((a + b)/2) \). Let \( N_2 \) be even, \( > N_1 \), and so large that \( \sum_{n>N_2} |a_{i_2,n}| < 1/4M \). Then any sequence \( \{c_n\} \) composed only of terms \( f(a), f(b), f((a + b)/2) \), whose terms from \( N_1 + 1 \) up to \( N_2 \) are all equal to \( f((a + b)/2) \), has the property that \( \sum_{n=1}^{N_2} a_{i_2,n}c_n \) differs in absolute value from \( f((a + b)/2) \sum_{n=1}^{N_2} a_{i_2,n} \) by less than \( M \cdot 1/2M + 2M \cdot 1/4M = 1 \). We shall
now find \( i_3 > i_2 \) and so large that \( \sum_{n=1}^{N_2} |a_{i_2,n}| < 1/4M \). Now choose \( N_3 \) even, \( > N_2 \), and so large that \( \sum_{n>N_3} |a_{i_3,n}| < 1/8M \). We now leave in our sequence \( \{c_n\} \) that we are constructing, terms \( a, b, a, b, \cdots \) starting from \( N_2 + 1 \) and stopping at \( N_3 \). Any sequence \( \{c_n\} \) with such terms in the indicated positions and its remaining terms chosen from among \( f(a), f(b), f((a+b)/2) \), has the property that \( \sum_{n=1}^{\infty} a_{i_3,n} c_n \) differs in absolute value from \( f(a) \sum_{n \text{ odd}} a_{i_3,n} + f(b) \sum_{n \text{ even}} a_{i_3,n} \) by less than \( M \cdot 1/4M + 2M \cdot 1/8M = 1/2 \). Choose \( i_4 > i_3 \) and so large that \( \sum_{n=1}^{N_4} |a_{i_4,n}| < 1/4M \). Then choose \( N_4 \) even, \( > N_3 \), and so large that \( \sum_{n>N_4} |a_{i_4,n}| < 1/8M \). Now interpolate \( f((a+b)/2) \) from the \( N_3 + 1 \) to the \( N_4 \) position. Any sequence \( \{c_n\} \) with \( f((a+b)/2) \) in these positions and its remaining terms chosen from among \( f(a), f(b), f((a+b)/2) \), has the property that \( \sum_{n=1}^{\infty} a_{i_4,n} c_n \) differs in absolute value from \( f((a+b)/2) \sum_{n=1}^{\infty} a_{i_4,n} \) by less than \( M \cdot 1/4M + 2M \cdot 1/8M = 1/2 \). Continuing in this fashion, we finally obtain a sequence \( \{c_n\} \) with the property that

\[
\left| \sum_{n=1}^{\infty} a_{i_{2k-1},n} c_n - f(a) \sum_{n \text{ odd}} a_{i_{2k-1},n} - f(b) \sum_{n \text{ even}} a_{i_{2k-1},n} \right| < \frac{1}{k},
\]

and

\[
\left| \sum_{n=1}^{\infty} a_{i_{2k},n} c_n - f\left(\frac{a+b}{2}\right) \sum_{n=1}^{\infty} a_{i_{2k},n} \right| < \frac{1}{k},
\]

Given any \( \epsilon > 0 \), if \( k > 2/\epsilon \) and also so large that

\[
\left| \sum_{n \text{ odd}} a_{i_{2k-1},n} - r \right| < \frac{\epsilon}{4M}, \quad \left| \sum_{n \text{ even}} a_{i_{2k-1},n} - (1-r) \right| < \frac{\epsilon}{4M},
\]

\[
\left| \sum_{n=1}^{\infty} a_{i_{2k},n} - 1 \right| < \frac{\epsilon}{2M},
\]

then we have

\[
\left| \sum_{n=1}^{\infty} a_{i_{2k-1},n} c_n - rf(a) - (1-r)f(b) \right| < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon,
\]

\[
\left| \sum_{n=1}^{\infty} a_{i_{2k},n} c_n - f\left(\frac{a+b}{2}\right) \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

From this, we conclude that the subsequence of the auxiliary means of \( \{c_n\} \) corresponding to those rows of \( A \) indexed by \( i_{2k-1}, k = 1, 2, \cdots \), converges to \( rf(a)+(1-r)f(b) \). Similarly, the subsequence of the auxiliary means of \( \{c_n\} \) under \( A \) corresponding to those rows of \( A \).
indexed by $i_{2k}$, $k = 1, 2, \cdots$, converges to $f((a+b)/2)$. Since a convergent sequence has but one limit point, we conclude that $f((a+b)/2) = rf(a) + (1-r)f(b)$. Starting with the sequence $b$, $a$, $b$, $a$, $\cdots$, we likewise conclude that $f((b+a)/2) = rf(b) + (1-r)f(a)$. Thus for all $a$ and $b$, $rf(a) + (1-r)f(b) = rf(b) + (1-r)f(a)$. Since $f$ is nonconstant, we can choose $a$, $b$ such that $f(a) \neq f(b)$. Then $r(f(a) - f(b)) = (1-r)(f(a) - f(b))$. Thus $r = 1 - r$, or $r = 1/2$. Then $f((a+b)/2) = (1/2)f(a) + (1 - 1/2)f(b)$, that is, $f((a+b)/2) = (f(a) + f(b))/2$ for all $a$ and $b$. Since $f$ is continuous, we conclude that $f$ is linear.

To prove the last part of the theorem, let $f(z) = cz + d$ with $c \neq 0$. The hypothesis of the theorem tells us that $\{cz_n + d\}$ is $A$-summable whenever $\{x_n\}$ is a bounded $(C, 1)$-summable sequence. Subtracting the sequence $d, d, d, \cdots$ from this sequence and dividing by $c$, we find that $\{x_n\}$ is $A$-summable whenever $\{x_n\}$ is a bounded $(C, 1)$-summable sequence. Theorem 1 of [3] is now exactly what one needs to conclude that $A$ sums every bounded $(C, 1)$-summable sequence to its $(C, 1)$ sum. This concludes the proof of the theorem.

The theorem is of course false for summability methods (as opposed to $(C, 1)$) whose convergence field is too small.

A question which arises in connection with the theorem has been answered by Professor H. Hanani: Namely, if the Toeplitz matrix $A$ sums every bounded $(C, 1)$-summable sequence, does it sum every $(C, 1)$-summable sequence? The answer is “no.” For let $a_{in} = 1/i$, $1 \leq n \leq i$, $a_{i,i} = 1/i$, $a_{in} = 0$ otherwise. The matrix $A = (a_{in})$ is a Toeplitz matrix which sums every bounded $(C, 1)$-summable sequence, as is easy to see. But the sequence $\{-1\}^*n^{1/2}$, $n = 1, 2, \cdots$, is $(C, 1)$-summable (to zero), whereas the sequence of its odd auxiliary means under $A$ converges to $+1$, its even ones to $-1$. Another question whose answer is “no” is this: if the Toeplitz matrix $A$ sums every bounded $(C, 1)$-summable sequence, does it give the $(C, 1)$ sum for any $(C, 1)$-summable sequence which it happens to sum? For let $A = (a_{in})$ with $a_{in} = 1/i$, $1 \leq n \leq i$, $a_{i,i} = (-1)^{i+1}1/i$, $a_{in} = 0$ otherwise. Then the same sequence as used above is summable by this matrix to $1$, not zero.

**Bibliography**


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