ON THE FINITE DIMENSIONALITY OF EVERY
IRREDUCIBLE UNITARY REPRESENTATION
OF A COMPACT GROUP

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We shall prove that every irreducible unitary representation of a compact group is finite dimensional. Our argument is a variation of known proofs and it hardly could be based on an idea different from those already current. It makes no use of the Peter-Weyl theorem and of compact or Hilbert-Schmidt operators and seems simpler than the proofs in [1; 2; 3; 4]. Its crucial point is that the prospectively finite dimension of the representation Hilbert space is expressible by a known integral formula.

Let \( \mathcal{H} \neq 0 \) be a Hilbert space and \( x \mapsto U_x \) be a group homomorphism of a compact group \( G \) into the group \( \mathcal{U}(\mathcal{H}) \) of all unitary operators in \( \mathcal{H} \), such that the scalar product \( (\xi | U_x \eta) \) is a continuous function of \( x \in G \) for all \( \xi, \eta \in \mathcal{H} \). Suppose that this representation is irreducible, namely that there is no closed vector subspace of \( \mathcal{H} \) invariant under all \( U_x \) except the trivial ones 0 and \( \mathcal{H} \). Then there results that \( \mathcal{H} \) is finite dimensional. In fact, let \( \xi, \eta, \xi', \eta' \in \mathcal{H} \). Denoting complex conjugation by a star,

\[
\int (\xi | U_x \eta) \cdot (\xi' | U_x \eta')^* \, dx \leq ||\xi|| \cdot ||\eta|| \cdot ||\xi'|| \cdot ||\eta'||,
\]

there is an operator \( T \) on \( \mathcal{H} \) depending on \( \eta, \eta' \) such that

\[
\int (\xi | U_x \eta) \cdot (\xi' | U_x \eta')^* \, dx = (T_x | \xi').
\]

\( T \) commutes with every \( U_x \) since

\[
(T U_x \xi | \xi') = \int (\xi | U_{x^{-1}} \eta) \cdot (\xi' | U_x \eta')^* \, dx
\]

\[
= \int (\xi | U_x \eta) \cdot (\xi' | U_{ix} \eta')^* \, dx = (T \xi | U_x \xi') = (U_x T \xi | \xi'),
\]

from which \( T U_x = U_x T \) follows. The irreducibility of the representation then implies that \( T \) is a scalar operator, that is \( T = \lambda(\eta, \eta') I \) and we get

\[
\int (\xi | U_x \eta) \cdot (\xi' | U_x \eta')^* \, dx = \lambda(\eta, \eta')(\xi | \xi').
\]

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By interchanging the roles of the couples \((\xi, \xi')\) and \((\eta, \eta')\) and using the rule \(\int f(x^{-1})dx = \int f(x)dx\), we get
\[
\lambda(\eta, \eta')(\xi | \xi') = \lambda(\xi, \xi')^*(\eta | \eta')^*.
\]
Hence \(\lambda(\eta, \eta') = c(\eta | \eta')^*\), where \(c\) is a constant, and
\[
(1) \quad \int (\xi, U_\xi\eta) \cdot (\xi' | U_\xi\eta')^* dx = c(\xi | \xi') \cdot (\eta | \eta')^*.
\]
If we let \(\xi, \eta, \xi', \eta'\) all become equal to a unit vector \(\alpha\), we get
\[
c = \int |(\alpha | U_\alpha\alpha)|^2 dx.
\]
Hence \(c > 0\), since the positive continuous function whose integral is \(c\) has strictly positive value at the identity of \(G\).

Now let \(e_1, \ldots, e_n\) be orthonormal in \(\mathcal{C}\). Let \(\eta, \eta'\) become equal to \(e_t\) and \(\xi, \xi'\) become equal to \(\alpha\) in (1). By adding the resulting equalities and using Bessel's inequality
\[
(2) \quad \sum_{i=1}^{n} (\alpha | U_\xi e_i) \cdot (\alpha | U_\xi e_i)^* \leq ||\alpha||^2 = 1
\]
since \(U_\xi e_1, \ldots, U_\xi e_n\) are orthonormal, we get \(nc \leq 1\), that is \(n \leq 1/c\).
This completes the proof that the dimension of \(\mathcal{C}\) is finite.

We remark that, if \(n\) is supposed to be the finite dimension of \(\mathcal{C}\), then (2) holds as an equality and so we get \(nc = 1\), that is \(c = 1/n\).
Then (1) becomes a known formula (see [5, Chapter V]) which we took as motivation for the above proof.

References

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