A SET IS A 3 CELL IF ITS CARTESIAN PRODUCT WITH AN ARC IS A 4 CELL

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We are learning strange things about cartesian products.

Shapiro has shown that the cartesian product of a line and a certain example described by Whitehead [16] is topologically $E^4$. The example is a bounded open subset of $E^3$ that is simply connected but which contains a simple closed curve which does not lie in any topological cube in it [3]. Glimm has given a proof of this result in [9] and a different proof is suggested in [5].

Artin and Fox have given [8] an example of a wild arc in $S^3$ whose complement is simply connected though topologically different from $E^3$. Kirkor has shown [11] that the cartesian product of this complement and a line is $E^4$.

Bing has described a nonmanifold [2] whose cartesian product with a line is $E^4$ [4; 5]. A modification of this example gives a compact nonmanifold whose cartesian product with a circle is topologically the same as the cartesian product of a 3 sphere and a circle. The modification is a monotone decomposition of $S^3$ whose set of nondegenerate elements is a Cantor set of tame arcs.

Rosen has complicated the example of Bing to show [14] that there is a set which is not locally a manifold at any point but whose cartesian product with a line is $E^4$. A proof that the cartesian product of a line and such a decomposition is $E^4$ has also been given by K. W. Kwun.

Fox and Artin described [8] an arc in $E^3$ whose complement is not simply connected. Curtis has shown [7] that the cartesian product of a line and the decomposition of $E^3$ whose only nondegenerate element is this arc is $E^4$. He later generalized this result to show that any decomposition of $E^n$ whose only nondegenerate element is an arc gives $E^{n+1}$ under a cartesian product with a line. One might wonder about any monotone decomposition of $E^3$ each of whose nondegenerate elements is an arc.

Poénaru gives [12] an example of a 4 manifold with boundary different from a cell whose cartesian product with an arc is a 5 cell. One might wonder just how far these unexpected things go.

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Question. Is there a complex different from a manifold whose cartesian product with a line is a manifold? The existence of such a complex would show that there is a triangulation of a manifold such that the star of a vertex is not a cell.

Raymond has shown [13] that $A \times B = M$ is a generalized manifold with possible boundary over a coefficient domain $L$ if and only if $A$ and $B$ are generalized manifolds over $L$.

The following theorem shows another direction in which the pathologies are limited.

Theorem. A set is a 3 cell if its cartesian product with an interval is a 4 cell.

Proof. Suppose $X \times [0, 1] = I^4$. We give a sequence of lemmas exhibiting enough properties of $X$ to insure that it is a 3 cell.

It follows from Lemma 1 that $X$ can be embedded in $E^3$. Hence, we suppose that $X$ lies in $E^3$ and use $\text{Bd } X$ and $\text{Int } X$ to denote the point set boundary and the interior respectively of the set $X$ in $E^3$.

We show that $\text{Bd } X$ is a 2 sphere. This follows from the Kline sphere characterization [1] since $\text{Bd } X$ is connected (Lemma 9), locally connected (Lemma 10) and is not separated by any pair of its points (Lemma 12) but is separated by each simple closed curve in it (Lemma 13). Dyer has pointed out to me that instead of using the Kline sphere characterization, one can use results of Wilder [17; 18] instead.

Since $\text{Int } X$ is uniformly locally simply connected (Lemma 11), $\text{Bd } X$ is tame from the inside [6]. This insures that $X$ is a 3 cell.

Suppose $M$ is a set whose cartesian product with an interval $[0, 1]$ is an $n$ cell $I^n$. In the following lemmas we note certain properties of $M$.

Lemma 1. $M$ can be embedded in $E^{n-1}$.

Proof. We show that each of $M \times 0$ and $M \times 1$ is contained in the $(n-1)$ sphere $\text{Bd } I^n$. Suppose $p \in M$ and $0 < \varepsilon < 1$. If $p \times \varepsilon$ were a point of the interior of the $n$ cell $M \times [0, \varepsilon]$, then $(p \times [\varepsilon, 1]) + (M \times [0, \varepsilon])$ would be an $n$ cell with a feeler sticking out of its interior. The invariance of domain theorem [10, p. 96] implies that such a set cannot be imbedded in $I^n$. Hence $p \times \varepsilon$ is a point of $\text{Bd}(M \times [0, \varepsilon])$. Since there is a homeomorphism of $M \times [0, \varepsilon]$ onto $M \times [0, 1]$ taking $p \times \varepsilon$ onto $p \times 1$, $p \times 1$ is a point of $\text{Bd } I^n$. Similarly $p \times 0$ is a point of $\text{Bd } I^n$.

Since $M$ can be imbedded in $E^{n-1}$, we suppose that it lies there. In the following lemmas we used $\text{Bd } M$ and $\text{Int } M$ to denote the point set boundary and interior respectively of $M$ in $E^{n-1}$.
Lemma 2. M is of dimension $n-1$ at each of its points.

Proof. It is not of dimension more than $n-1$ at any point since it lies in $E^{n-1}$. It is not of dimension less than $n-1$ at any point or else $I^n = M \times [0, 1]$ would be of dimension less than $n$ somewhere.

Lemma 3. M is the closure of an open subset of $E^{n-1}$.

Proof. If it were not such a subset, it would be of dimension less than $n-1$ at some point.

Lemma 4. $\text{Bd } M \times (0, 1) \subset \text{Bd } I^n$.

Proof. Suppose $p \in \text{Bd } M$ and $N$ is a spherical neighborhood of $p$ in $E^{n-1}$. Then $p \times \epsilon$ $(0 < \epsilon < 1)$ is a limit point of $(N \times (0, 1)) - (M \times (0, 1))$. It follows from the invariance of domain that $p \times \epsilon$ does not lie in any open $n$ manifold in $M \times (0, 1)$. Hence $p \times \epsilon$ is not a point of $\text{Int } I^n$.

Lemma 5. $\text{Int } M \times (0, 1) \subset \text{Int } I^n$.

This result follows from the invariance of domain theorem. We summarize Lemmas 1, 4, 5 as follows:

\[
\text{Bd } I^n = (M \times 0) + (M \times 1) + (\text{Bd } M \times (0, 1)),
\]

\[
\text{Int } I^n = \text{Int } M \times (0, 1).
\]

We recall certain definitions. Let $F_t$ $(0 \leq t \leq 1)$ be a one parameter (the parameter is $t$) family of maps of a set $A$ into $B$ such that if $F_t$ is regarded as taking $A \times t$ into $B$, $F_t$ $(0 \leq t \leq 1)$ takes $A \times [0, 1]$ continuously into $B$. Then $F_t$ $(0 \leq t \leq 1)$ is called a homotopy pulling $F_0$ to $F_1$. We are particularly interested in the case where $F_t(A)$ is a point. In this case we say $F_t$ is a constant map.

A map $f$ of $A$ into $B$ is inessential if there is a homotopy of $A$ into $B$ pulling $f$ to a constant map. If there is no such homotopy, $f$ is called an essential map.

A subset $A$ of $B$ can be shrunk to a point in $B$ if the cone over $A$ can be mapped into $B$ in such a way that the map is the identity on the base of the cone. (This cone lies in an abstract space and is not supposed to intersect $B$ except at its base.) Hence, $A$ can be shrunk to a point in $B$ if the identity map of $A$ into $B$ is inessential.

A set is contractible if the identity map of the set into itself is inessential.

A set $A$ is uniformly locally connected in dimension $n$ if for each $\epsilon > 0$, there is a $\delta > 0$ such that each map of an $n$ sphere $S^n$ into a $\delta$ subset of $A$ can be extended to map an $(n+1)$-ball bounded by $S^n$.
into an $\epsilon$ subset of $A$. We say that the map of $S^n$ into a $\delta$ subset of $A$ can be shrunk to a constant on an $\epsilon$ subset of $A$.

In determining whether or not $M$ is uniformly locally connected it is irrelevant which metric we use for $M$ since $M$ is compact. However, there is more of a problem when we deal with Int $M$. We suppose that $M$ is given a prescribed metric and that this metric is the same on Int $M$. We suppose that the metric of $M \times [0, 1] = I^n$ is given by the metric of a cube in $E^n$ rather than by the cartesian metric.

**Lemma 6.** $M$ is uniformly locally connected in all dimensions.

**Proof.** This is because $M \times [0, 1]$ has these properties. To see that each map $f$ of an $m$ sphere $S^m$ into a small subset of $M$ can be shrunk to a constant map on a small subset of $M$, consider the map $f'$ of $S^m$ into $M \times 1/2$ given by $f'(p) = f(p) \times 1/2$. Consider a homotopy $F_t (0 \leq t \leq 1)$ shrinking $f'$ to a constant in a small subset of $I^n$. Let $g$ be the projection of $I^n$ onto $M$ that takes $p \times \epsilon$ to $p$. Then $gF_t (0 \leq t \leq 1)$ is a homotopy shrinking the map $f$ to a constant on a small subset of $M$.

**Lemma 7.** $M$ is contractible.

The proof of Lemma 7 is similar to the proof of Lemma 6.

**Lemma 8.** $\text{Bd } M \times 1/2$ irreducibly separates $M \times 0$ from $M \times 1$ on $\text{Bd } I^n$.

**Proof.** Each point $p \times 1/2$ of $\text{Bd } M \times 1/2$ is needed to separate $M \times 0$ from $M \times 1$ on $\text{Bd } I^n$ since $p \times [0, 1]$ is an arc on $\text{Bd } I^n$ from $M \times 0$ to $M \times 1$.

**Lemma 9.** $\text{Bd } M$ is connected.

**Proof.** This follows from Lemma 8 and the unicoherence of the $(n-1)$ sphere $\text{Bd } I^n$.

**Lemma 10.** $\text{Bd } M$ is uniformly locally connected in all dimensions.

**Proof.** The proof is the same as the proof of Lemma 6 except that we define $F_t$ to be a homotopy of $S^n$ into $\text{Bd } I^n - ((M \times 0) + (M \times 1))$.

**Remark.** We do not prove that $\text{Bd } M$ is homotopically trivial even in low dimensions. In fact Poénaru's example [12] shows that $\text{Bd } M$ need not even be simply connected in the case where $n = 5$.

**Lemma 11.** $\text{Int } M$ is uniformly locally connected in all dimensions.

**Proof.** We are to prove that for each $\epsilon > 0$ there is a $\delta > 0$ such that each map of an $m$ sphere $S^m$ into a $\delta$ subset of $\text{Int } M$ can be shrunk
to a constant map on an \( \varepsilon \) subset of \( \text{Int} \ M \).

Let \( h \) be the homeomorphism of \( M \) onto \( M \times 1/2 \) given by \( h(p) = p \times 1/2 \) and \( g \) the projection of \( I^n \) onto \( M \times 1/2 \) given by \( g(p \times t) = p \times 1/2 \). We suppose \( I^n \) has its ordinary Euclidean metric. Let \( \varepsilon > 0 \) be so small that each \( \varepsilon \) subset of \( I^n \) has an image under \( h^{-1}g \) of diameter less than \( \varepsilon \). Let \( \delta > 0 \) be so small that each \( \delta \) subset of \( M \) has a diameter of less than \( \varepsilon \) under \( h \).

Let \( f \) be a map of \( S^m \) into a \( \delta \) subset of \( \text{Int} \ M \). Then \( hf \) takes \( S^m \) into an \( \varepsilon \) subset of \( \text{Int} \ I^n \). There is a homotopy \( F_t \ (0 \leq t \leq 1) \) shrinking \( hf \) to a constant map on an \( \varepsilon \) subset of \( \text{Int} \ I^n \). Then \( h^{-1}gF_thf \) shrinks \( f \) to a constant map on an \( \varepsilon \) subset of \( \text{Int} \ M \).

**Lemma 12.** No \((n - 4)\) sphere in \( \text{Bd} \ M \) separates it.

**Proof.** Suppose \( T \) is a topological \((n - 4)\) sphere in \( \text{Bd} \ M \). Then \( T \times (0, 1) \) is an \((n - 3)\) manifold in \( \text{Bd} \ M \times (0, 1) \). However, \( \text{Bd} \ M \times (0, 1) \) is a connected open subset of \( \text{Bd} \ I^n \) and is hence a connected \((n - 1)\) manifold. Since no \((n - 3)\) manifold even locally separates an \((n - 1)\) manifold, \( T \) does not separate \( \text{Bd} \ M \).

**Lemma 13.** Each topological \((n - 3)\) sphere \( T \) in \( \text{Bd} \ M \) separates it.

**Proof.** Consider \((M \times 0) + (M \times 1) + (T \times (0, 1)) = X \). We show in Lemma 14 that there is an essential map of \( X \) onto an \((n - 2)\) sphere. This shows \([10]\) that \( X \) separates the \((n - 1)\) sphere \( \text{Bd} \ I^n \) and that \( \text{Bd} \ I^n - X = (\text{Bd} \ M - T) \times (0, 1) \) is not connected. Hence, \( \text{Bd} \ M - T \) is not connected.

Lemma 13 can also be proved by using the methods of homology theory by showing that \((M \times 0) + (M \times 1) + (T \times (0, 1)) \) contains an \((n - 2)\) Čech cycle that does not bound in \((M \times 0) + (M \times 1) + (T \times (0, 1)) \).

The proof of the following lemma was supplied by a conversation with E. E. Floyd and Walter Rudin.

**Lemma 14.** Suppose \( A_1, A_2 \) are two closed sets whose intersection is an \( n \) sphere \( S^n \) such that \( S^n \) can be shrunk to a point in each \( A_i \). Then there is an essential map of \( A_1 + A_2 \) onto an \((n + 1)\) sphere.

**Proof.** Let \( C_1, C_2 \) be two \((n + 1)\) cells sewed together along their boundaries to form an \((n + 1)\) sphere. Let \( f \) be a map of \( A_1 + A_2 \) into \( C_1 + C_2 \) that takes the \( n \) sphere \( A_1 \cdot A_2 \) homeomorphically onto \( \text{Bd} \ C_1 = \text{Bd} \ C_2 = C_1 \cdot C_2, A_1 \) into \( C_1 \), and \( A_2 \) into \( C_2 \).

We show that \( f \) is essential by showing that there is no homotopy \( F_t \ (0 \leq t \leq 1) \) of \( A_1 + A_2 \) into \( C_1 + C_2 \) that shrinks \( f \) to a constant map.
We show that the assumption that there is such a homotopy \( F_t \) leads to the contradiction that the identity map of \( C_1 + C_2 \) onto itself is inessential.

Since \( A_1 \cdot A_2 \) can be shrunk to a point in \( A_1 \), there is a map \( g \) of \( C_1 \) into \( A_1 \) that is \( f^{-1} \) in taking \( C_1 \cdot C_2 \) onto \( A_1 \cdot A_2 \). Similarly, the map \( g \) can be extended to take \( C_2 \) into \( A_2 \). We call the extended map \( g \).

Consider the map \( fg \). It is homotopic to the identity since it takes each \( C_i \) \( (i = 1, 2) \) onto itself and is the identity on \( \text{Bd} \ C_1 = \text{Bd} \ C_2 \). The assumption that \( f \) is inessential leads to the contradiction that \( F_t g (0 \leq t \leq 1) \) is a homotopy pulling \( fg \) to a constant map.

It may be noted that the proof of Lemma 14 did not use the fact that \( A_1 \cdot A_2 \) is an \( n \) sphere but merely that \( A_1 \cdot A_2 \) contains an \( n \) sphere that is a retract of \( A_1 \cdot A_2 \).

A homological analogue of Lemma 14 gives that if \( A_1, A_2 \) are compact and \( A_1 \cdot A_2 \) contains an \( n \) cycle which bounds in each of \( A_1, A_2 \) but not in \( A_1 \cdot A_2 \), then \( A_1 + A_2 \) contains an \((n + 1)\) cycle that does not bound in \( A_1 + A_2 \).

**Theorem.** Each factor of an \( n \) cell \( I^n, n \leq 4 \), is a cell.

**Proof.** Szumbarski proved [15] this result for \( n \leq 3 \).

Szumbarski also proved [15] that each 1 dimensional factor of an \( n \) cell is a 1 cell.

Young showed [19] that each 2 dimensional factor of an \( n \) cell is a 2 cell.

Young's proof of Theorem 1 in [19] showed that if one factor of an \( n \) cell is of dimension \( n - 1 \), the other factor contains no triod. Hence, if a factor of \( I^4 \) is 3 dimensional, the other factor is an arc and it follows from our first theorem that the 3 dimensional factor is a 3 cell.

**References**

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