ON THE REALIZABILITY OF SINGULAR COHOMOLOGY GROUPS

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Let $H_n(X)$ and $H^n(X)$ be the integral singular homology and cohomology groups of a space $X$ and let $\mathcal{G}$ be the category of abelian groups. Then it is well known that for every sequence $(A_1, A_2, \ldots, A_n, \ldots)$ with the $A_n \in \mathcal{G}$, there exists a space $X$ such that $H_n(X) \approx A_n$ for all $n > 0$. We will show that the analogous statement for cohomology is false. In fact we prove:

**Theorem.** There exists no space $X$ and integer $n \geq 1$ such that $H^{n-1}(X) = 0$ and $H^n(X) \approx \mathbb{Z}_0$ (the additive group of the rationals).

In the proof the following results will be used.

(a) $\mathbb{Z}_0$ has no nontrivial direct sum decomposition (trivial).

(b) $	ext{Hom}(A, \mathbb{Z})$ is not divisible for any $A \in \mathcal{G}$ (trivial).

(c) Let $A \in \mathcal{G}$ and $\text{Hom}(A, \mathbb{Z}) = 0$. Then $\text{Ext}(A, \mathbb{Z})$ is divisible if and only if $A$ is torsionfree and $\text{Ext}(A, \mathbb{Z})$ is torsionfree if and only if $A$ is divisible.

**Proof.** We will write $\text{Hom} B$ and $\text{Ext} B$ instead of $\text{Hom}(B, \mathbb{Z})$ and $\text{Ext}(B, \mathbb{Z})$. For any integer $m > 1$ consider the exact sequence

$$0 \to mA \to A \to A \to A_m \to 0.$$ 

Because $\text{Hom} A = \text{Hom} mA = 0$ application of the functor $\text{Ext}$ yields the exact sequence

$$0 \to \text{Ext} A_m \to \text{Ext} A \to \text{Ext} A \to \text{Ext} mA \to 0$$

and hence $\text{Ext} A_m = m(\text{Ext} A)$ and $\text{Ext} mA = (\text{Ext} A)_m$. For any torsion group $T$, $\text{Ext} T = 0$ if and only if $T = 0$. Hence $mA = 0$ if and only if $\text{Ext} mA = m(\text{Ext} A) = 0$. The proposition now follows from the fact that a group $B \in \mathcal{G}$ is torsionfree if and only if $mB = 0$ for all $m > 1$ and that $B$ is divisible if and only if $B_m = 0$ for all $m > 1$.

(d) If $A \in \mathcal{G}$ is torsionfree and divisible, then $A$ is a vector space over $\mathbb{Z}_0$ (trivial).

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2 The first half of this proposition was proved by R. J. Nunke (Illinois J. Math. vol. 3 (1959) p. 230) without the restriction $\text{Hom}(A, \mathbb{Z}) = 0$. 

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(e) If \( j: A \to B \in \mathcal{G} \) is a monomorphism, then \( \text{Ext}(j, Z): \text{Ext}(B, Z) \to \text{Ext}(A, Z) \) is an epimorphism (trivial).

(f) \( \text{Ext}(Z_0, Z) \) is not countable.

**Proof.** The exact sequence \( 0 \to Z \to Z_0 \to Z_0/Z \to 0 \) induces an exact sequence

\[
0 \to \text{Hom}(Z_0, Z_0) \to \text{Hom}(Z_0, Z_0/Z) \to \text{Ext}(Z_0, Z) \to 0.
\]

As \( \text{Hom}(Z_0, Z_0) \approx Z_0 \) is countable it suffices to show that \( \text{Hom}(Z_0, Z_0/Z) \) is not. For every sequence \( a_1, a_2, \ldots, a_n, \ldots \in Z_0/Z \) such that \( na_n = a_{n-1} \) for all \( n \) there clearly is a homomorphism \( f: Z_0 \to Z_0/Z \) such that \( f(1/n!) = a_n \). As the set of these sequences is not countable neither is \( \text{Hom}(Z_0, Z_0/Z) \).

**Proof of the theorem.** Let \( X \) be a space such that \( H^{n-1}(X) = 0 \) and \( H^n(X) \approx Z_0 \). By the universal coefficient theorem

\[
0 = H^{n-1}(X) \approx \text{Hom}(H^{n-1}(X), Z) + \text{Ext}(H_{n-2}(X), Z)
\]

\[
Z_0 \approx H^n(X) \approx \text{Hom}(H^n(X), Z) + \text{Ext}(H_{n-1}(X), Z).
\]

Hence \( \text{Hom}(H_{n-1}(X), Z) = 0 \), by (a) and (b) \( Z_0 \approx \text{Ext}(H_{n-1}(X), Z) \) and thus, by (c) \( H_{n-1}(X) \) is torsionfree and divisible. But then (d), (e) and (f) imply that \( \text{Ext}(H_{n-1}(X), Z) \) is not countable which is a contradiction, q.e.d.

**Remark.** It is not known whether in the theorem the hypothesis \( H^{n-1}(X) = 0 \) can be omitted, i.e., whether \( Z_0 \) can be a singular cohomology group at all.

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