NOTE ON A TOPOLOGY OF A DUAL SPACE

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1. Introduction. Let $G$ be a locally compact group, and let $L$ be a Lie group. Let us denote by $D$ the set of all continuous homomorphisms from $G$ into $L$. By introducing the so-called compact-open topology in $D$, $D$ becomes a complete uniform space. We shall call the uniform space $D$ the dual space of $G$ with respect to $L$ in this paper.

When $G$ is abelian and $L$ is the group of rotations in the euclidean plane, $D$ coincides with the character group in the sense of Pontrjagin, and it is well known that $D$ is also locally compact. Here we would like to generalize the proposition to the nonabelian case. The purpose of this note is to prove the following two theorems:

**Theorem 1.** Let $G$ be a locally compact group, and $L$ a compact Lie group. Then the dual space $D$ is locally compact.

**Theorem 2.** Let $G$ be a locally compact group and $L$ a Lie group. If there is a compact generating system in $G$, then the dual space $D$ is locally compact.

**Remark.** If $G$ is an infinitely generated free group with the discrete topology, and if $L$ is a noncompact Lie group, then $D$ is homeomorphic with an infinite product of copies of $L$ and accordingly $D$ is not locally compact.

2. Preliminaries. Let $M$ be a set and $H$ a topological group. Let us denote by $(M \to H)$ the topological space composed of all functions from $M$ into $H$ relative to the product topology (= finite-open topology). By defining the multiplication: $(fg)(x) = f(x)g(x)$ for $f, g \in (M \to H)$ and $x \in M$, $(M \to H)$ becomes a topological group. When $M$ is a topological space, we consider a subset (subgroup) $[M \to H]$ of $(M \to H)$ composed of all continuous functions. For a compact subset $C$ of $M$ and a neighborhood $E$ of the identity in $H$ we define a subset

$$(C, E) = \{f \mid f \in [M \to H] \text{ and } f(C) \subset E\} \text{ of } [M \to H].$$

It is easy to see that the set of all possible $(C, E)$'s forms a base for a neighborhood system of the identity of a topological group. After

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2 For the first two sections please refer to the books listed in the end of the paper.
this \([M\rightarrow H]\) denotes the topological group thus topologized. Clearly the identity mapping brings \([M\rightarrow H]\) continuously into \((M\rightarrow H)\).

Let us assume that \(M\) is a topological group. Let \(D'\) be the set of all homomorphisms from \(M\) into \(H\). We define \(D\) by \(D = D' \cap \[M\rightarrow H\].\) Since an element \(f\) of \(D'\) is characterized by the property \(f(x)f(y) = f(xy)\) for any pair \(x, y\) of \(M\), \(D'\) is a closed subspace of \((M\rightarrow H)\).

\(D\) is closed in \([M\rightarrow H]\) similarly. When \(H\) is a complete topological group \((M\rightarrow H)\) and \([M\rightarrow H]\) are known to be complete, and so \(D'\) and \(D\) are complete. Hence \(D'\) and \(D\) are complete uniform spaces. It is to be noted that when \(H\) is locally compact \(f(x)\) is a jointly continuous function from \([M\rightarrow H]\times M\) into \(H\).

**Proposition 1.** Let \(G\) and \(L\) be locally compact groups. Then (1) \((G\rightarrow L)\) and \([G\rightarrow L]\) are complete topological groups, and \(D'\) and \(D\) are closed subspaces of \((G\rightarrow L)\) and \([G\rightarrow L]\) respectively. (2) For an \(f\) in \([G\rightarrow L]\) \(f(x)\) is jointly continuous from \([G\rightarrow L]\times G\) into \(L\). (3) When \(L\) is compact, \(D'\) is compact.

**Notations.** We shall write 1 for the identity of the group in question. By a nucleus of a locally compact group we mean an open symmetric neighborhood of 1, whose closure is compact. Let \(G\) and \(L\) be locally compact groups, and \(D\) the dual space of \(G\) with respect to \(L\). Let \(f\) be in \(D\). For a compact subset \(C\) of \(G\) and a nucleus \(E\) of \(L\) we put

\[(f; C, E) = \{g \mid g \in D \text{ and } (f^{-1}g)(C) \subseteq E\}.\]

The set of all possible \((f; C, E)\)'s of course forms a base for the neighborhood system of \(f\) in \(D\).

Let \(L\) be a Lie group of dimension \(r\). Let us introduce a canonical system of coordinates of the first kind in a suitable nucleus of \(L\). By changing the scale if necessary, we may identify a nucleus \(E\) with an open euclidean sphere of radius 2: for \(a\) in \(E\), there corresponds coordinates \((a_1, \ldots, a_r)\) so that \(\|a\|^2 = a_1^2 + \cdots + a_r^2 \leq 2\). We denote \(S(\delta) = \{a \mid \|a\| < \delta\}\) for \(0 < \delta \leq 2\). By the definition of a canonical coordinate system of the first kind, \(ta = (ta_1, \ldots, ta_r)\) forms a local one-parameter subgroup for \(-2/\|a\| < t < 2/\|a\|\), and if \(j\) is a positive integer less than \(2/\|a\|\), then \(a^j = ja\).

**Proposition 2.** Let \(m\) be a positive integer. Then an element \(a\) of \(L\) is contained in the sphere \(S(1/m)\) if and only if \(a, a^2, \ldots, a^m \in S(1)\).

**Proof.** If \(a \in S(1/m)\), namely if \(\|a\| < 1/m\), then for \(j \leq m, j < 1/\|a\| < 2/\|a\|\), and so \(a^j = ja\) and \(\|a^j\| = j\|a\| < 1\).

Conversely if \(a\) is in \(S(1)\) but not in \(S(1/m)\), namely if \(1/m \leq \|a\|\)
<1, and if \( j \) is the first integer so that \( 1 \leq j \|a\| \), then \( j \leq m \) and \( j \|a\| < 2 \), whence \( a^j = ja \in S(1) \).

3. Proof of Theorem 1. Let \( G \) be a locally compact group, and \( L \) a compact Lie group. We retain the notations in 2. We know that \((G \to L)\) is compact, and the identity mapping \( I \) is continuous and one-to-one from \( D \) into \((G \to L)\). Let \( f \) be an element of \( D \). Since \( f \) is continuous we can find a nucleus \( V \) of \( G \) so that \( f(V) \subseteq S(1/2) \). Take a nucleus \( E \) of \( L \) so that \( S(1/2) \overline{E} \subseteq S(1) \), where \( \overline{E} \) denotes the closure of \( E \). Let us prove that the neighborhood \((f; \overline{V}, E)\) has a compact closure in \( D \). Let us put \( F = \{g \in D \text{ and } (f^{-1}g)(\overline{V}) \subseteq \overline{E}\} \). It clearly suffices to prove that

\begin{enumerate}
  \item \( I \) is an open mapping in \( F \), and
  \item \( I(F) \) is closed in \((G \to L)\).
\end{enumerate}

Proof of (A). Let \( g \) be in \( F \), and let \((g; C_1, E_1)\) be a given neighborhood of \( g \). Let us find a finite subset \( \{x_1, \ldots, x_k\} \) of \( G \) and a positive integer \( m \) so that \((g; \{x_1, \ldots, x_k\}, S(1/m)) \subseteq C_1 \). For this purpose take a sufficiently large \( m \) such that \( S(1/m) \subseteq E_1 \), and take a nucleus \( U \) of \( G \) such that \( U \subseteq V \). Since \( C_1 \) is compact we can find a finite subset \( \{x_1, \ldots, x_k\} \) of \( G \) so that \( x_1 U \cup \cdots \cup x_k U \subseteq C_1 \). We shall prove that \( \{x_1, \ldots, x_k\} \) and \( m \) thus obtained satisfy the requirement.

Let us take an \( h \) in \((g; \{x_1, \ldots, x_k\}, S(1/m)) \cap F \). Let \( x \) be in \( C_1 \). Then \( x \) can be written in a form \( x = x_j y \) where \( y \in U \) and \( 1 \leq j \leq k \). Hence \( g(x_j)^{-1}h(x_j) = (g(x_j)g(y))^{-1}h(x_j)h(y) = g(y)^{-1}(g(x_j)^{-1}h(x_j))h(y) \).

Since \( h \) is in \((g; \{x_1, \ldots, x_k\}, S(1/m)) \), we have
\[
g(x_j)^{-1}h(x_j) \in S(1/m).
\]

Next for \( n = 1, 2, \ldots, m \), \( y^n \in V \) and so \( f^{-1}(y^n)g(y^n) \subseteq \overline{E} \), and \( f(y^n) \subseteq S(1/2) \). Hence \( g(y^n) \subseteq S(1/2) \overline{E} \subseteq S(1) \). Because \( g(y^n) = g(y)^n \) we have \( g(y) \subseteq S(1/m) \) by Proposition 2. Similarly \( h(y) \subseteq S(1/m) \).

Hence \( g(x_i)^{-1}h(x_i) \subseteq S(1/m) \subseteq E_1 \), namely \( h \in (g; C_1, E_1) \).

Proof of (B). Let \( \text{Cl}(I(F)) \) be the closure of \( I(F) \) in \((G \to L)\). Since \( D' \) is closed in \((G \to L)\) we have \( \text{Cl}(I(F)) \subseteq D' \). Let \( g \) be an element of \( \text{Cl}(I(F)) \) and let \( E_2 \) be a given nucleus of \( L \). We may take \( n \) so large that \( S(1/n) \subseteq E_2 \). Take a nucleus \( W \) of \( G \) so that \( W^{n+1} \subseteq V \). Then for an \( x \) in \( W \), \( x, x^2, \ldots, x^{n+1} \in V \) and so \( f(x^i)^{-1}h(x^i) \subseteq \overline{E} \) for \( h \in F \) and \( i = 1, 2, \ldots, n+1 \). Therefore \( h(x^i) = h(x)^i \subseteq S(1/2) \overline{E} \subseteq S(1) \). Hence by Proposition 2, \( h(x) \subseteq S(1/(n+1)) \). Since \( g \in \text{Cl}(I(F)) \), \( g(x) \in \text{Cl}(S(1/(n+1))) \subseteq S(1/n) \subseteq E_2 \), namely \( g(W) \subseteq E_2 \), which implies that \( g \subseteq I(D) \). Thus we proved that \( \text{Cl}(I(F)) \subseteq I(D) \), and so \( I(F) \) is closed.
4. Proof of Theorem 2. Let $G$ be a locally compact group with a compact generating system. Then obviously there is a nucleus $V$ which generates $G$. Let us consider a function $J$ from $D$ into $[V \rightarrow L]$ defined by $Jf = f|\overline{V}$, where $f|\overline{V}$ is the restriction of $f$ in $\overline{V}$.

**Lemma 1.** $J$ is bicontinuous, namely $D$ is homeomorphic with $J(D)$.

**Proof.** Since $V$ generates $G$, $J$ is one-to-one. The continuity is obvious. Let us prove the openness of $J$.

Let $(f; C_1, E_1)$ be a given neighborhood of $f$ in $D$. We can find a positive integer $m$ such that $V^m \supset C_1$. Let us consider a function $\phi$ defined by

$$\phi(u_1, \ldots, u_m; a_2, \ldots, a_m) = a_m^{-1}(\ldots (a_2^{-1}u_1a_2)u_2a_3u_3)\ldots)a_mu_m,$$

where $u_i \in L$ and $a_i \in f(\overline{V})$. Since $\phi(1, \ldots, 1; a_2, \ldots, a_m) = 1$, we can find a nucleus $E_2$ of $L$ so that if $u_1, \ldots, u_m$ are in $E_2$, then $\phi(u_1, \ldots, u_m, a_2, \ldots, a_m)$ is in $E_1$. Let $x$ be an arbitrary element in $C_1$. Then we can find $x_1, \ldots, x_m$ in $V$ so that $x = x_1 \cdots x_m$. Let $g$ be in $(f; V, E_2)$. Then

$$f(x)^{-1}g(x) = (f(x_1) \cdots f(x_m))^{-1}(g(x_1) \cdots g(x_m)) = \phi(u_1, \ldots, u_m; a_2, \ldots, a_m)$$

where $f(x_j)^{-1}g(x_j) = u_j \in E_2$ and $f(x_j) = a_j \in f(\overline{V})$. Hence $f(x)^{-1}g(x) \in E_1$, namely $(f; V, E_2) \subseteq (f; C_1, E_1)$, which proves the lemma.

Next let us fix an element $f$ of $D$, and for a $g$ in $D$ we define $g^*$ in $[G \rightarrow L]$ by $g^*(x) = f(x)^{-1}g(x)$. Then $g^*$ satisfies

$$(1) \quad g^*(xy) = f(y)^{-1}g^*(x)f(y)g^*(y)$$

for $x$ and $y$ in $G$, and conversely if a function $g^*$ in $[G \rightarrow L]$ satisfies (1), then $g^* = f^{-1}g$ for some $g$ in $D$. Let $W$ be an arbitrary nucleus of $G$. Set

$$F = f^{-1}(f; \overline{V}, \text{Cl}(S(1/2))) = \{g^* \mid g^*(\overline{W}) \subseteq \text{Cl}(S(1/2)) \text{ and } g^* \in f^{-1}D\}.$$ 

**Lemma 2.** $F|\overline{W} = \{g^* \mid g^*(xy) \subseteq E \}$ is an equicontinuous family of functions.

**Proof.** Let $E$ be a given nucleus of $L$. Let us prove the existence of a nucleus $U$ of $G$ so that $x, xy \in \overline{W}$, $y \in U$ and $g^* \in F$ imply that $g^*(x)^{-1}g^*(xy) \subseteq E$. For this purpose let us first take an integer $m$, with $S(1/m) \subseteq E$, and a positive number $\epsilon$ such that

$$(2) \quad \text{Cl}(S(1/2))S(\epsilon)^2 \subseteq S(1).$$
Next for the $\epsilon$ and the $m$ we shall take a positive number $\delta_1$ such that if $a, b \in S(1)$ and $c \in S(\delta_1)$ then

$$f(U) \subseteq S(\delta) \quad \text{and} \quad U^m \subseteq W.$$  

Next let us find a positive number $\delta_2$ such that if $a \in S(1)$ and $c \in S(\delta_2)$ then

$$g*(\mathcal{X}) = (\mathcal{X})^{-1}g*(\mathcal{X}) = g*(\mathcal{X})^{-1}f(\mathcal{X})^{-1}g*(\mathcal{X})f(\mathcal{X})g*(\mathcal{X}).$$

Hence using (3) we have

$$(g*(\mathcal{X})^{-1}g*(\mathcal{X}))^i \in g*(\mathcal{X})^iS(\epsilon) \quad \text{for } j = 1, 2, \cdots, m.$$  

On the other hand, since

$$g*(\mathcal{X}) = (f(\mathcal{X})^{-1}g*(\mathcal{X})f(\mathcal{X}))^i \cdots (f(\mathcal{X})^{-1}g*(\mathcal{X})f(\mathcal{X}))g*(\mathcal{X}),$$

(4) implies that

$$g*(\mathcal{X}) \subseteq g*(\mathcal{X})^iS(\epsilon).$$  

From (7) and (8) we have

$$(g*(\mathcal{X})^{-1}g*(\mathcal{X}))^i \subseteq g*(\mathcal{X})S(\epsilon)^2.$$  

From $y^i \in W$ it follows that $g*(y^i) \subseteq Cl(S(1/2))$, whence

$$(g*(\mathcal{X})^{-1}g*(\mathcal{X}))^i \subseteq Cl(S(1/2))S(\epsilon)^2 \subseteq S(1)$$

by (2), for $j = 1, 2, \cdots, m$. Hence by Proposition 2

$$g*(\mathcal{X})^{-1}g*(\mathcal{X}) \subseteq S(1/m) \subseteq E.$$  

**Lemma 3.** $F \mid W$ is compact, if $W$ generates $G$.

**Proof.** We shall denote by $(\mathcal{W} \to Cl(S(1/2)))$ the topological space composed of all functions from $\mathcal{W}$ into $Cl(S(1/2))$ with the product topology, and let us consider the identity mapping $I$ from $F \mid \mathcal{W}$ into $(\mathcal{W} \to Cl(S(1/2)))$, which is clearly one-to-one and continuous. Because $(\mathcal{W} \to Cl(S(1/2)))$ is compact, it suffices to prove that $I$ is an open mapping and $I(F \mid \mathcal{W})$ is a closed set.

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(A) Openness of $I$. Let $g^*$ be an element of $F$. For a neighborhood of $g^*$ in $F$, we can find a smaller one of the form $(g^*; W, E) = \{ h^* | h^* \in f^{-1}D \text{ and } (g^*^{-1}h^*)(\overline{W}) \subset E \}$, where $E$ is a nucleus of $L$. Let us find a finite subset $\{ x_1, \ldots, x_k \}$ of $\overline{W}$ and a nucleus $E_1$ of $L$ such that $(g^*; W, E) \supset (g^*; \{ x_1, \ldots, x_k \}, E_1) \cap F$, where $(g^*; \{ x_1, \ldots, x_k \}, E_1) = f^{-1}(g; \{ x_1, \ldots, x_k \}, E_1)$.

For this purpose let us first take a nucleus $E_1$ of $L$ satisfying $E_1a^{-1}E_1aE_1 \subset E$ for $a \in f(\overline{W})$, and find a nucleus $U$ of $G$ in $W$ so that $F(U) \subset E_1$ using Lemma 2. Next let us take $x_1, \ldots, x_k \in \overline{W}$ so that $x_i \in U \cup \cdots \cup x_k U \supset W$.

Let $x$ be in $\overline{W}$. Then we can find a $y$ in $U$ such that $x = x_j y$ for some $j$. Let $h^*$ be in $(g^*; \{ x_1, \ldots, x_k \}, E_1) \cap F$. Then by (1) $g^*(x) = g^*(x_jy) = g^*(y)$ since $y \in U$. On the other hand, since $y \in U$ we have $g^*(y) \in E_1$ and $h^*(y) \in E_1$, and $h^* \in (g^*; \{ x_1, \ldots, x_k \}, E_1)$ implies that $g^*(x_j) = h^*(x_j) \in E_1$. From $U \subset \overline{W}$ it follows that $f(U) \subset f(\overline{W})$. Hence $g^*(x) \in f(\overline{W})$. Therefore $g^*(x) \in f(\overline{W})$$\subset E$.

(B) $I(F|\overline{W})$ is closed.

For a pair $x, y$ of elements of $\overline{W}$ so that $xy$ is also in $\overline{W}$, and for a $g^*$ in $F$ we have the relation (1). Hence if $h^*$ is in the closure of the image of $F$ in $(\overline{W} \rightarrow S(1/2))$, then $h^*(xy) = f(y)^{-1}h^*(x)f(y)h^*(y)$. Let us put $h(x) = f(x)h^*(x)$ for $x \in \overline{W}$. Then $h(xy) = h(x)h(y)$ for $x, y, xy \in \overline{W}$. Since $F|\overline{W}$ is equicontinuous by Lemma 2, $h^*$ is continuous, and so is $h$.

Let $x_1, \ldots, x_m, y_1, \ldots, y_n$ be elements of $\overline{W}$. If $x_1 \cdots x_m = y_1 \cdots y_n$, then for $g = fg^* \in fF$, we have $g(x_1) \cdots g(x_m) = g(y_1) \cdots g(y_n)$, whence we have $h(x_1) \cdots h(x_m) = h(y_1) \cdots h(y_n)$.

Because $\overline{W}$ generates $G$, $h$ can be extended to a continuous homomorphism $\overline{h}$ from $G$ into $L$.

Hence $h^* = f^{-1}\overline{h} \in F$, and $\overline{h} = h^*$ in $\overline{W}$. Therefore $I(F|\overline{W})$ is closed.

Proof of Theorem 2. By Lemma 1, $D$ is homeomorphic with $J(D)$. On the other hand if we put $W = V$ in Lemma 3 then we have the result that $J(F)$ is compact. Since $[\overline{V} \rightarrow L]$ is a topological group, $J(f)J(F) = J(f; \overline{V}, \text{Cl}(S(1/2)))$ is also compact. Accordingly $J(D)$ is locally compact, and so is $D$.

References

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