ON HOMOTHETIC MAPPINGS OF RIEMANN SPACES

M. S. KNEBELMAN AND K. YANO

The object of this note is to generalize some results obtained in [1; 2] and to give a shorter proof to one of them.

1. Let V_n be a Riemann space with fundamental metric $g_{ij}(x)$. Let $\xi^i(x)$ be a vector field defining a one-parameter Lie group and L the symbol of Lie differentiation based on $\xi^i(x)$. The condition that $\xi^i(x)$ define a motion, an affine collineation, a homothetic transformation or a conformal transformation of V_n is

$$Lg_{ij} = \xi_{i,j} + \xi_{j,i} = 0, \tag{1.1}$$

$$L\begin{Bmatrix} i \\ jk \end{Bmatrix} = \xi^i_{,j,k} + R^i_{jkl}\xi^l = 0, \tag{1.2}$$

$$Lg_{ij} = 2cg_{ij}, \tag{1.3}$$

or

$$Lg_{ij} = 2\phi g_{ij} \tag{1.4}$$

respectively, where $\xi_{i,j}$ is the covariant derivative of ξ_i with respect to the Christoffel symbols $\left\{ {i \atop jk} \right\}$ and R^i_{jkl} the curvature tensor of V_n, c and ϕ being a constant and a function of x respectively. When c vanishes, a homothetic transformation reduces to a motion. Thus we call a proper homothetic transformation one for which $c \neq 0$.

Since we have the formulas

$$L\begin{Bmatrix} i \\ jk \end{Bmatrix} = \frac{1}{2} g^{ia}[(Lg_{ja})_{,k} + (Lg_{ak})_{,j} - (Lg_{jk})_{,a}], \tag{1.5}$$

$$\left(L\begin{Bmatrix} i \\ jk \end{Bmatrix}\right)_{,l} - \left(L\begin{Bmatrix} i \\ jl \end{Bmatrix}\right)_{,k} = LR^i_{jkl}, \tag{1.6}$$

it is easily seen that a motion and a homothetic transformation are both affine collineations and that an affine collineation preserves the curvature tensor.

In [1] one of the present authors proved that in a space of nonzero constant curvature a mapping preserving curvature is a motion. For an Einstein space with nonzero curvature scalar, a mapping preserving Ricci curvature is a motion; for applying the operator L to

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$$R_{ij} = \alpha g_{ij}, \qquad \alpha = \text{constant}, \ n > 2,$$

we have

$$0 = LR_{ij} = \alpha L g_{ij},$$

from which, α being different from zero, we have

$$Lg_{ij} = 0.$$

Since a homothetic transformation is an affine collineation, we have, as a corollary of this:

THEOREM 1. *In an Einstein space of nonzero curvature scalar, a homothetic transformation is a motion.*

This generalizes Theorem 2.2 of [2].

For a homothetic transformation, we have

$$\begin{aligned} LR &= L(g^{ij}R_{ij}) = (Lg^{ij})R_{ij} \\ &= -2cg^{ij}R_{ij} = -2cR, \end{aligned}$$

where R is the curvature scalar. Thus if $R = \text{constant} \neq 0$, then we have $c = 0$, from which we have

THEOREM 2. *In a Riemann space with nonzero constant curvature scalar, a homothetic transformation is a motion.*

This generalizes Theorem 1.

For a conformal transformation, we have

$$\phi_{,i,j} = LC_{ij}$$

where

$$C_{ij} = -\frac{R_{ij}}{n-2} + \frac{Rg_{ij}}{2(n-1)(n-2)}, \qquad n > 2.$$

thus if R is a constant, we have

$$g^{ij}\phi_{,i,j} = -\frac{R}{n-1}\phi, \tag{1.7}$$

from which we conclude

THEOREM 3. *If a space with constant curvature scalar admits a proper homothetic transformation, then the curvature scalar vanishes.*

In the case where the space is compact and orientable, one of the present authors [3] proved that an affine collineation is a motion and so a homothetic transformation is also a motion.

In a compact orientable Riemann space with constant curvature scalar, substituting (1.7) into a famous integral formula

$$\frac{1}{2}\int g^{ij}(\phi^2)_{,i,j}\, d\sigma = \int (\phi g^{ij}\phi_{,i,j} + g^{ij}\phi_{,i}\,\phi_{,j})d\sigma = 0$$

we find

$$\int \left(-\frac{R}{n-1}\phi^2 + g^{ij}\phi_{,i}\,\phi_{,j}\right) d\sigma = 0.$$

Thus if $R<0$, then $\phi=0$ and if $R=0$, then $\phi=$constant. When $\phi=0$ the conformal transformation is a motion and when $\phi=$constant, the conformal transformation is a homothetic transformation and is consequently an affine collineation. Thus corresponding to Theorem 2, we have

THEOREM 4. *In a compact orientable Riemann space with constant curvature scalar* <0, *a conformal transformation is a motion.*

2. Assume that the Riemann space V_n admits a group G_{r+1} of homothetic transformations and denote by $\xi^i_{(\alpha)}$ $(\alpha, \beta = 1, 2, \cdots, r, r+1)$ generators of the group and by L_α the operators of Lie differentiation with respect to $\xi^i_{(\alpha)}$. Then we have

$$L_\alpha g_{ij} = 2c_\alpha g_{ij}, \tag{2.1}$$

where c's are constants not all zero. Without loss of generality we can assume that

$$c_a = 0 \qquad (a = 1, 2, \cdots, r). \tag{2.2}$$

Now the following relation holds good [4]:

$$(L_\alpha L_\beta - L_\beta L_\alpha)g_{ij} = \overset{\gamma}{c}_{\alpha\beta} L_\gamma g_{ij}, \tag{2.3}$$

where c's are constants of structure.

Putting $\alpha=a$, $\beta=r+1$ in (2.3), we find

$$\overset{r+1}{c}_{a\,r+1} = 0 \tag{2.4}$$

by virtue of (2.1) and (2.2). Equation (2.4) proves Theorem 2.1 in [2], that is,

The full group G_{r+1} of homothetic transformations of V_n contains an invariant subgroup G_r of motions and a G_1 subgroup of dilations.

3. **Finsler spaces.** In such a space arc length is defined by $ds^2 = F(x, dx)$ homogeneous of degree 2 in dx. In such a space covariant

differentiation may be defined in different ways but in any case it is not based on the Christoffel symbols.

As for Riemann spaces a motion is defined by

$$Lg_{ij} = \xi^h g_{ij,h} + g_{ih}\xi^h_{,j} + g_{hj}\xi^h_{,i} + g_{ij,h}\xi^h_{,k}dx^k = 0 \tag{3.1}$$

where $g_{ij,h} = \partial g_{ij}/\partial(dx^h)$, while

$$Lg_{ij} = 2\phi g_{ij} \tag{3.2}$$

defines a homothetic mapping if ϕ is a constant and a conformal mapping if it is not a constant (ϕ is necessarily independent of dx). We showed above that a homothetic mapping in a Riemann space is an affine collineation and the converse, i.e., a conformal mapping which is an affine collineation is homothetic. To prove this for a Finsler space we use the fact that for a conformal mapping

$$L\Gamma^i_{jk} = \delta^i_j\phi_{,k} + \delta^i_k\phi_{,j} - \frac{1}{2}(Fg^{ih})_{,j,k}\phi_{,h}. \tag{3.3}$$

Obviously if $\phi = \text{constant}$ $L\Gamma^i_{jk} = 0$ which defines an affine collineation. To prove the converse we have

$$\delta^i_j\phi_{,k} + \delta^i_k\phi_{,j} - \frac{1}{2}(Fg^{ih})_{,j,k}\phi_{,h} = 0 \tag{3.4}$$

so that

$$2dx^i\phi_{,k}dx^k - F(g^{ih}\phi_{,h}) = 0. \tag{3.5}$$

Contracting the above with $g_{ij,l}$ we find $Fg^{ih}g_{ij,l}\phi_{,h} = 0$ and since $g^{ih}g_{ij,l} = -g_{ij}g^{th}_{,l}$ so that $(g^{ih}\phi_{,h})_{,l} = 0$.

Differentiating (3.5) partially with respect to dx^l we obtain

$$\delta^i_l\phi_{,k}dx^k + \phi_{,l}dx^i - g_{jl}dx^j g^{ih}\phi_{,h} = 0$$

and contraction gives $n\phi_{,k}dx^k = 0$ and hence $\phi = \text{constant}$.

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WASHINGTON STATE UNIVERSITY AND
UNIVERSITY OF HONG KONG