INNER PRODUCT SPACES AND THE TRI-SPHERICAL INTERSECTION PROPERTY

W. W. COMFORT AND HUGH GORDON

1. Introduction and definitions. This note adds another criterion to the lengthy list of those properties which characterize the real inner product spaces whose dimension exceeds two. For a concise guide to the history of the development of this subject, the reader is referred to Day [2, VII, §3].

1.1. Definition. A subset \( F \) of a linear space \( E \) is said to be a flat if, for some \( x \in E \), the set \( F - x \) is a linear subspace of \( E \).

1.2. Definition. Let \( x_1, x_2 \) and \( x_3 \) be points in a normed linear space \( E \). Then \( F(x_1, x_2, x_3) \) denotes the smallest flat containing each of the points \( x_i \).

1.3. Definition. If \( E \) is a normed linear space and \( x \in E \) and \( \rho > 0 \), then by \( S_{\rho}(x) \) we mean the set \( \{ y \in E | ||x - y|| \leq \rho \} \).

1.4. Definition. A normed linear space \( E \) is said to have the tri-spherical intersection property if, whenever \( x_1, x_2 \) and \( x_3 \) are points in \( E \) and \( \rho_1, \rho_2 \) and \( \rho_3 \) are positive numbers with \( \rho_1 < \rho_2 < \rho_3 \), it follows that \( \bigcap_{i=1}^{3} S_{\rho_i}(x_i) \neq \emptyset \).

1.5. Remark. Every two-dimensional real normed linear space has the tri-spherical intersection property. Every real normed linear space has the bi-spherical intersection property, defined by analogy with 1.4.

2. Theorems from the literature. We list here three theorems upon which the proof of our result depends.

2.1. Definition. (Birkhoff. See [1, p. 169].) If \( x \) and \( y \) are elements of a real normed linear space \( E \), then we say \( y \) is orthogonal to \( x \), and write \( y \perp x \), if \( ||y - \alpha x|| \geq ||y|| \) for each real number \( \alpha \). If \( J \) is a subset of \( E \), then we write \( J \perp x \) if \( y \perp x \) for each \( y \in J \).

2.2. Theorem. (James. See [5, Theorem 4].) A real normed linear space whose dimension exceeds 2 is an inner product space provided that for each hyperplane \( H \) of \( E \) containing \( \phi \) there is a point \( x \) of \( E \) for which \( x \neq \phi \) and \( H \perp x \).

2.3. Theorem. (Fréchet. See [3, p. 717].) In order that a real
normed linear space be an inner product space, it is sufficient that each of its three-dimensional subspaces be an inner product space.

2.4. Theorem. (Helly. See [4].) Let $D$ be an $n$-dimensional real normed linear space, and let $\mathcal{F}$ be a collection of compact, convex subsets of $D$. If every $n+1$ elements of $\mathcal{F}$ have a point in common, then some point of $D$ lies in every element of $\mathcal{F}$.

3. Characterization of inner product space.

3.1. Theorem. Let $E$ be a real normed linear space of dimension $\geq 3$. Then the following assertions are equivalent:

(a) $E$ is an inner product space,
(b) $E$ has the tri-spherical intersection property.

Proof. (a)$\Rightarrow$(b). Let $x_1$, $x_2$ and $x_3$ be points in $E$, and let $z \in \bigcap_{i=1}^{3} S_{\rho_i}(x_i)$ for certain positive numbers $\rho_i$. The set $F(x_1, x_2, x_3)$, being complete in $E$, is closed in the completion $\overline{E}$ of $E$. Clearly we may suppose that $\phi \in F(x_1, x_2, x_3)$. Let $P$ denote the orthogonal projection of $\overline{E}$ onto $F(x_1, x_2, x_3)$. Then either each $x_i = \phi$ or $\|P\| = 1$, and in any event we have $\|x_i - P(x)\| = \|P(x_i - z)\| \leq \|x_i - z\| \leq \rho_i$ for each $i$. Thus $P(z) \in \bigcap_{i=1}^{3} S_{\rho_i}(x_i)$.

(b)$\Rightarrow$(a). Let $D$ be a three-dimensional linear subspace of $E$, and let $H$ be a two-dimensional linear subspace of $D$. In view of 2.2 and 2.3, we need only show that $H \perp x$ for some $x \in D$ with $x \neq \phi$. Choose any $z \in D$ which is not in $H$. Define $\mathcal{F} = \{S_{\rho}(y) | \rho > 0, y \in H$ and $\|y - z\| \leq \rho\}$. Since $D$ inherits the tri-spherical intersection property from $E$, every three elements of $\mathcal{F}$ have a point in common which belongs to $H$. Hence by 2.4 there is a point $w \in H$ with the property that if $y \in H$ and $\|y - z\| \leq \rho$, then $\|y - w\| \leq \rho$, i.e., with the property that if $y \in H$, then $\|y - w\| \leq \|y - z\|$. Now define $x = z - w$. Since $z \in H$ and $w \in H$, we have $x \neq \phi$. Then for each nonzero scalar $\alpha$ and each $y \in H$ we have

$$\|y/\alpha - x\| = \|(y/\alpha + w) - z\| \geq \|(y/\alpha + w) - w\| = \|y/\alpha\|.$$ 

Hence $\|y - \alpha x\| \geq \|y\|$ and $H \perp x$.

3.2. Remark. From 2.4 with $n = 2$ it follows that a real normed linear space $E$ has the tri-spherical intersection property if and only if the following statement is true: If $x_1$, $x_2$ and $x_3$ are points in $E$ and $\rho_1$, $\rho_2$ and $\rho_3$ are positive numbers with $\bigcap_{i=1}^{3} S_{\rho_i}(x_i) \neq \emptyset$, then $\bigcap_{i=1}^{3} S_{\rho_i}(x_i) \cap \text{conv}(x_1, x_2, x_3) \neq \emptyset$. (By $\text{conv}(x_1, x_2, x_3)$ we mean the smallest convex set containing each of the points $x_i$.)

3.3. Remark. If the tri-spherical intersection property had been defined in 1.4 using open spheres of the form
S_\rho(x) = \{ y \in E \mid \| x - y \| < \rho \},

then the statement of 3.1 remains valid. A minor modification of the proof of the implication (b) \implies (a) is required.

4. Characterization of ellipsoids. Let E be a Euclidean space. Let S be a compact convex body in E symmetric about one of its points. We call this point the centre of S. To avoid complications, we shall define an ellipsoid in E to be a convex body which determines a norm given by an inner product. We call a subset of E homothetic to S if it is of the form \alpha S + x with \alpha > 0 and x \in E. Our result now takes the following form: S is not an ellipsoid if and only if there are sets S_1, S_2 and S_3, with centres x_1, x_2 and x_3, homothetic to S, such that S_1 \cap S_2 \cap S_3 \neq \emptyset but S_1 \cap S_2 \cap S_3 \cap \text{conv}(x_1, x_2, x_3) = \emptyset.

References


Harvard University