ON A CLASS OF UNIMODAL DISTRIBUTIONS

R. G. LAHA

A distribution function \( F(x) \) is said to be unimodal \([1, \text{p. 157}]\) if there exists at least one value \( x = a \) such that \( F(x) \) is convex for \( x < a \) and concave for \( x > a \). The point \( x = a \) is called the vertex of the distribution. In particular, if \( F(x) \) is absolutely continuous, then the corresponding probability density function \( p(x) = F'(x) \) is nondecreasing for \( x < a \) and nonincreasing for \( x > a \). In the present paper we shall establish the unimodality of a class of distribution functions which were studied by Linnik in \([2]\). He has shown that for any real \( \alpha \) in the interval \( 0 < \alpha \leq 2 \) the function

\[
f(t) = \frac{1}{1 + |t|^\alpha}
\]

is the characteristic function of a symmetric and absolutely continuous distribution function. We denote this class of distribution functions by \( C \). We shall now prove the following theorem.

**Theorem. Every distribution function belonging to the class \( C \) is unimodal.**

For the proof of this theorem we give first the following lemma which may be of some independent interest.

**Lemma 1.** Let \( f(t) \) be a continuous real-valued and even function of the real variable \( t \) such that \( f(0) = 1 \) and \( f(t) = A(t) \) for \( t > 0 \) where the function \( A(t) \) satisfies the following conditions:

(i) \( A(z) \) defined as a function of the complex variable \( z \) (\( z = re^{it} = t + iv, t \) and \( v \) both real) is regular in the region \( D \) \((r > 0; -e_1 < \theta < \pi/2 + e_2 \) where \( e_1 \) and \( e_2 \) are arbitrary small positive numbers) of the complex \( z \)-plane;

(ii) \[ |A(z)| = O(1) \quad \text{as} \quad |z| \to 0, \]

(iii) \[ |A(z)| = O(|z|^{-\delta}) \quad \text{as} \quad |z| \to \infty \quad (\delta > 1); \]

\( \text{Im} \ A(iv) \leq 0 \quad \text{for} \ v > 0. \)

Then \( f(t) \) is the characteristic function of a symmetric unimodal and absolutely continuous distribution function.

Received by the editors January 18, 1960 and, in revised form, May 2, 1960.

1 This work was supported in part by The National Science Foundation Grant NSF-G-4220 and in part by the Office of Naval Research Contract NR-042-064.
Proof. First we note that for real values of \( t \)

\[
|f(t)| = O(1) \quad \text{as} \quad |t| \to 0,
\]

\[
|f(t)| = O(|t|^{-\delta}) \quad \text{as} \quad |t| \to \infty, \quad (\delta > 1),
\]

so that \( f(t) \) is absolutely integrable over the interval \( (-\infty < t < +\infty) \). Therefore for any real \( x \) the function

\[
\rho(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx}f(t)\,dt = \frac{1}{\pi} \int_{0}^{\infty} \cos tx \, f(t)\,dt
\]

exists and is a continuous, real-valued and even function of \( x \) so that we have \( \rho(x) = \rho(-x) \) for \( x \geq 0 \). We shall now show that \( \rho(x) \geq 0 \) for \( x \geq 0 \). We see easily from (1) that for \( x \geq 0 \)

\[
\rho(x) = \frac{1}{\pi} \text{Re} \int_{0}^{\infty} e^{itx}f(t)\,dt = \frac{1}{\pi} \text{Re} \int_{0}^{\infty} e^{itx}A(t)\,dt.
\]

In order to evaluate the integral on the right-hand side of (2), we consider the function \( \psi(z) = e^{itz}A(z) \) (\( z \) complex and \( x \geq 0 \)) and a closed contour \( \Gamma \) in the first quadrant of the complex \( z \)-plane (\( z = t + iv \), \( t \) and \( v \) both real) consisting of the real axis from \( \rho \) to \( R \), the larger circular arc of radius \( R \), the imaginary axis from \( iR \) to \( iv \) and finally the smaller arc of radius \( \rho \). The function \( \psi(z) \) is regular in the contour \( \Gamma \). Therefore, according to the theorem of Cauchy we have

\[
\int_{\Gamma} e^{itz}A(z)\,dz = 0
\]

or

\[
\int_{\rho}^{R} e^{itz}A(t)\,dt + I + \int_{R}^{\rho} e^{-vx}A(iv)\,dv + J = 0.
\]

Here \( I \) and \( J \) denote respectively the integrals along the circular arcs of radii \( R \) and \( \rho \). We can easily verify that \( I \to 0 \) as \( R \to \infty \) and similarly \( J \to 0 \) as \( \rho \to 0 \). Therefore as \( \rho \to 0 \) and \( R \to \infty \), we obtain from (3)

\[
\int_{0}^{\infty} e^{itz}A(t)\,dt = i \int_{0}^{\infty} e^{-vx}A(iv)\,dv.
\]

Then we conclude easily from (2) and (4) that for \( x \geq 0 \)

\[
\rho(x) = -\frac{1}{\pi} \int_{0}^{\infty} e^{-vz}[\text{Im } A(iv)]\,dv.
\]

The non-negativity of \( \rho(x) \) for \( x \geq 0 \) follows immediately from the condition (iii) of the lemma. Finally we show that for \( x \geq 0 \), the
probability density function $p(x)$ is a decreasing function of $x$. From (5) we see that for $x_1 > x_2 > 0$,

$$p(x_1) < p(x_2) < p(0)$$

so that the probability density function $p(x)$ has a unique maximum at the point $x = 0$. This completes the proof of the lemma.

**Proof of the Theorem.** For the proof of the theorem, we shall discuss separately three mutually exclusive possible cases:

**Case 1** ($0 < \alpha \leq 1$). In this case we consider the function

$$\theta(t) = f(t) + tf'(t) = \frac{1 + (1 - \alpha) |t|^\alpha}{(1 + |t|^\alpha)^2}.$$

We can easily verify after some elementary computations that for $t > 0$

$$\theta''(t) = \frac{\alpha t^{\alpha-2}}{(1 + t^\alpha)^4} [(1 - \alpha^2) + 2(1 + 2\alpha^2)t^\alpha + (1 - \alpha^2)t^{2\alpha}] > 0.$$

Thus we conclude that the function $\theta(t)$ introduced in (6) is a continuous, real-valued and even function of the real variable $t$ and further it is convex for $t > 0$ and $\theta(0) = 1$. The function $\theta(t)$ satisfies all the conditions of Pólya's theorem [3, p. 116] and hence it is a characteristic function. Finally we note that the function $f(t)$ can be represented in the form

$$f(t) = \frac{1}{t} \int_0^t \theta(u) du$$

where $\theta(u)$ itself is a characteristic function. The unimodality of this class of distribution functions follows immediately from the theorem due to Khintchine [1, p. 160].

**Case 2** ($1 < \alpha < 2$). Here we define the function

$$A(z) = \frac{1}{1 + z^\alpha}$$

as a function of the complex variable $z$ and then verify easily that $A(z)$ satisfies the conditions (i), (ii) and (iii) of Lemma 1. Therefore the unimodality of this class follows as a direct consequence of this lemma.

**Case 3** ($\alpha = 2$). In this case it is well known that the probability density function of the corresponding distribution function is given by

$$p(x) = \frac{1}{2} e^{-|x|}, \quad (-\infty < x < \infty).$$
which has a unique maximum at the point $x = 0$. This completes the proof of the theorem. The following corollary is a direct consequence of this theorem.

**Corollary.** Every symmetric stable distribution function is unimodal.

This result has been already proved by Wintner [4, p. 33]. We give below a simple alternative proof. First we state two lemmas:

**Lemma 2.** The convolution of two symmetric unimodal distribution functions is symmetric unimodal.

This lemma is due to Wintner [4, p. 30].

**Lemma 3.** If a sequence of unimodal distribution functions converges to a distribution function, then the limiting distribution function is also unimodal.

The proof of this lemma is given in [1, p. 160].

We now turn to the proof of the corollary. First, we note that the characteristic function of a symmetric stable distribution function is given by $g(t) = e^{-|t|^\alpha}$ ($0 < \alpha \leq 2$) [1, p. 164]. We write $a_n = n^{-1/\alpha}$ where $n$ is a positive integer. Then we conclude from our theorem and Lemma 2 that for every positive integer $n$, the function

$$g_n(t) = [f(a_n t)]^n = \frac{1}{\left(1 + \frac{|t|^\alpha}{n}\right)^n}$$

is the characteristic function of a symmetric unimodal distribution function. The proof of the corollary follows immediately from Lévy's continuity theorem and Lemma 3.

**References**


**Catholic University and Cornell University**