ON A CLASS OF UNIMODAL DISTRIBUTIONS

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A distribution function $F(x)$ is said to be unimodal [1, p. 157] if there exists at least one value $x=a$ such that $F(x)$ is convex for $x<a$ and concave for $x>a$. The point $x=a$ is called the vertex of the distribution. In particular, if $F(x)$ is absolutely continuous, then the corresponding probability density function $p(x)=F'(x)$ is nondecreasing for $x<a$ and nonincreasing for $x>a$. In the present paper we shall establish the unimodality of a class of distribution functions which were studied by Linnik in [2]. He has shown that for any real α in the interval $0<\alpha\leqq 2$ the function

$$f(t) = \frac{1}{1+|t|^{\alpha}}$$

is the characteristic function of a symmetric and absolutely continuous distribution function. We denote this class of distribution functions by C. We shall now prove the following theorem.

THEOREM. *Every distribution function belonging to the class C is unimodal.*

For the proof of this theorem we give first the following lemma which may be of some independent interest.

LEMMA 1. *Let $f(t)$ be a continuous real-valued and even function of the real variable t such that $f(0)=1$ and $f(t)=A(t)$ for $t>0$ where the function $A(t)$ satisfies the following conditions:*

(i) *$A(z)$ defined as a function of the complex variable z ($z=re^{i\theta}=t+iv$, t and v both real) is regular in the region D ($r>0$; $-\epsilon_1<\theta<\pi/2+\epsilon_2$ where ϵ_1 and ϵ_2 are arbitrary small positive numbers) of the complex z-plane;*

(ii)
$$|A(z)| = O(1) \quad \text{as } |z| \to 0,$$
$$|A(z)| = O(|z|^{-\delta}) \quad \text{as } |z| \to \infty \quad (\delta > 1);$$

(iii)
$$\operatorname{Im} A(iv) \leqq 0 \quad \text{for } v > 0.$$

Then $f(t)$ is the characteristic function of a symmetric unimodal and absolutely continuous distribution function.

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PROOF. First we note that for real values of t

$$\begin{aligned} |f(t)| &= O(1) && \text{as } |t| \to 0, \\ |f(t)| &= O(|t|^{-\delta}) && \text{as } |t| \to \infty, \qquad (\delta > 1), \end{aligned}$$

so that $f(t)$ is absolutely integrable over the interval $(-\infty < t < +\infty)$. Therefore for any real x the function

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} f(t)dt = \frac{1}{\pi}\int_0^{\infty} \cos tx\, f(t)dt \tag{1}$$

exists and is a continuous, real-valued and even function of x so that we have $p(x) = p(-x)$ for $x \geqq 0$. We shall now show that $p(x) \geqq 0$ for $x \geqq 0$. We see easily from (1) that for $x \geqq 0$

$$p(x) = \frac{1}{\pi}\operatorname{Re}\int_0^{\infty} e^{itx} f(t)dt = \frac{1}{\pi}\operatorname{Re}\int_0^{\infty} e^{itx} A(t)dt. \tag{2}$$

In order to evaluate the integral on the right-hand side of (2), we consider the function $\psi(z) = e^{izx}A(z)$ (z complex and $x \geqq 0$) and a closed contour Γ in the first quadrant of the complex z-plane ($z = t + iv$, t and v both real) consisting of the real axis from ρ to R, the larger circular arc of radius R, the imaginary axis from iR to $i\rho$ and finally the smaller arc of radius ρ. The function $\psi(z)$ is regular in the contour Γ. Therefore, according to the theorem of Cauchy we have

$$\int_\Gamma e^{izx} A(z)dz = 0$$

or

$$\int_\rho^R e^{itx} A(t)dt + I + \int_R^\rho e^{-vx} A(iv)idv + J = 0. \tag{3}$$

Here I and J denote respectively the integrals along the circular arcs of radii R and ρ. We can easily verify that $I \to 0$ as $R \to \infty$ and similarly $J \to 0$ as $\rho \to 0$. Therefore as $\rho \to 0$ and $R \to \infty$, we obtain from (3)

$$\int_0^{\infty} e^{itx} A(t)dt = i\int_0^{\infty} e^{-vx} A(iv)dv. \tag{4}$$

Then we conclude easily from (2) and (4) that for $x \geqq 0$

$$p(x) = -\frac{1}{\pi}\int_0^{\infty} e^{-vx}[\operatorname{Im} A(iv)]dv. \tag{5}$$

The non-negativity of $p(x)$ for $x \geqq 0$ follows immediately from the condition (iii) of the lemma. Finally we show that for $x \geqq 0$, the

probability density function $p(x)$ is a decreasing function of x. From (5) we see that for $x_1 > x_2 > 0$,

$$p(x_1) < p(x_2) < p(0)$$

so that the probability density function $p(x)$ has a unique maximum at the point $x=0$. This completes the proof of the lemma.

PROOF OF THE THEOREM. For the proof of the theorem, we shall discuss separately three mutually exclusive possible cases:

CASE 1 $(0<\alpha\leqq 1)$. In this case we consider the function

$$\theta(t) = f(t) + tf'(t) = \frac{1 + (1-\alpha)\,|\,t\,|^{\alpha}}{(1 + |\,t\,|^{\alpha})^2}. \tag{6}$$

We can easily verify after some elementary computations that for $t>0$

$$\theta''(t) = \frac{\alpha t^{\alpha-2}}{(1+t^{\alpha})^4}\left[(1-\alpha^2) + 2(1+2\alpha^2)t^{\alpha} + (1-\alpha^2)t^{2\alpha}\right] > 0.$$

Thus we conclude that the function $\theta(t)$ introduced in (6) is a continuous, real-valued and even function of the real variable t and further it is convex for $t>0$ and $\theta(0)=1$. The function $\theta(t)$ satisfies all the conditions of Pólya's theorem [3, p. 116] and hence it is a characteristic function. Finally we note that the function $f(t)$ can be represented in the form

$$f(t) = \frac{1}{t}\int_0^t \theta(u)\,du$$

where $\theta(u)$ itself is a characteristic function. The unimodality of this class of distribution functions follows immediately from the theorem due to Khintchine [1, p. 160].

CASE 2 $(1<\alpha<2)$. Here we define the function

$$A(z) = \frac{1}{1+z^{\alpha}}$$

as a function of the complex variable z and then verify easily that $A(z)$ satisfies the conditions (i), (ii) and (iii) of Lemma 1. Therefore the unimodality of this class follows as a direct consequence of this lemma.

CASE 3 $(\alpha=2)$. In this case it is well known that the probability density function of the corresponding distribution function is given by

$$p(x) = \frac{1}{2}e^{-|x|}, \qquad (-\infty < x < \infty),$$

which has a unique maximum at the point $x=0$. This completes the proof of the theorem. The following corollary is a direct consequence of this theorem.

COROLLARY. *Every symmetric stable distribution function is unimodal.*

This result has been already proved by Wintner [**4**, p. 33]. We give below a simple alternative proof. First we state two lemmas:

LEMMA 2. *The convolution of two symmetric unimodal distribution functions is symmetric unimodal.*

This lemma is due to Wintner [**4**, p. 30].

LEMMA 3. *If a sequence of unimodal distribution functions converges to a distribution function, then the limiting distribution function is also unimodal.*

The proof of this lemma is given in [**1**, p. 160].

We now turn to the proof of the corollary. First, we note that the characteristic function of a symmetric stable distribution function is given by $g(t)=e^{-|t|^\alpha}$ $(0<\alpha\leqq 2)$ [**1**, p. 164]. We write $a_n=n^{-1/\alpha}$ where n is a positive integer. Then we conclude from our theorem and Lemma 2 that for every positive integer n, the function

$$g_n(t) = [f(a_n t)]^n = \frac{1}{\left(1+\dfrac{|t|^\alpha}{n}\right)^n}$$

is the characteristic function of a symmetric unimodal distribution function. The proof of the corollary follows immediately from Lévy's continuity theorem and Lemma 3.

REFERENCES

1. B. V. Gnedenko, and A. N. Kolmogorov, *Limit distributions for sums of independent random variables* (English translation by K. L. Chung), Reading, Massachusetts, Addison-Wesley Publishing Co., Inc., 1954.

2. Yu. V. Linnik, *Linear forms and statistical criteria. Part* 2 (in Russian), Ukrain. Mat. Ž. vol. 5 (1953) pp. 247–290.

3. G. Pólya, *Remarks on characteristic functions*, Proceedings of the First Berkeley Symposium, 1949, pp. 115–123.

4. A. Wintner, *Asymptotic distributions and infinite convolutions*, lecture notes, Institute for Advanced Study, 1938.

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