AXIOMATIZATION OF INDUCED THEORIES

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This note gives a method of presenting axiom systems for various induced theories like the natural numbers theory of the real numbers theory and the pure set theory (with no constants except $\in$ and $=$ and with set variables only) induced by the Zermelo-Fraenkel set theory with the axion of foundation, an operation symbol $\sigma(x)$, an axiom $x \neq 0 \supset \sigma(x) \in x$ and all the instances of the axiom schemata containing the symbol σ added. By the results of Novak [4] and Shoenfield [5] the latter set theory is, as far as sets, not classes, are concerned, the same as the system A, B, C, D, E of Gödel [3]. We shall use the terminology and notations of [6].

Let R be a first order predicate language (with identity, without predicate variables) and let Q be any theory with standard formalization. Let f be a relative interpretation of R in Q. f attaches to every sentence ϕ of R a sentence $f(\phi)$ of Q which is obtained from ϕ by relativizing the quantifiers in ϕ to a new predicate P and then replacing all the nonlogical constants in ϕ by their definitions in Q. We assume, in the definition of a relative interpretation, that[1] $Q \vdash f((\exists x)(x=x))$. Q/R—the theory induced by Q on R (by means of f)—is defined as follows: ϕ is a theorem of Q/R if $f(\phi)$ is a theorem of Q.

We assume the metalanguage of Q to be arithmetized.

THEOREM. *Let R be a language as described above and let f be a relative interpretation of R in a theory with standard formalization Q. Q is given by a specified set of axioms (not necessarily recursive). Let there exist a relative interpretation g of Peano's number theory P in the induced theory Q/R. The composite function fg gives an interpretation of P in Q. Let Q be essentially reflexive with respect to the interpretation fg of P, i.e., for every sentence*[2] *ϕ of Q $Q \vdash \phi \supset fg$ ("ϕ is consistent"). Under these conditions an axiom system for Q/R is given by the two following schemata:*

(a) *$g(\phi)$, where ϕ is any axiom of P;*

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[1] Some further simple assumptions are needed if R contains operation symbols.

[2] By "ϕ is provable" we mean that ϕ is provable from the axioms of logic. By "ϕ is consistent" we mean that $\sim\phi$ is not provable. Due to results of Montague [2] there is a large class of theories which are reflexive with respect to a fixed interpretation of number theory, e.g., Peano's number theory, real number theory and Zermelo set theory are essentially reflexive with respect to the usual interpretations of number theory in those theories.

(b) $g(\text{“}(\psi \supset f(\phi))$ *is provable*$\text{”}) \supset \phi$, *where* ψ *is any conjunction of axioms of* Q, *and* ϕ *any sentence of* R.[3]

PROOF. All the instances of (a) are theorems of Q/R by the assumption that g is a relative interpretation of P in Q/R. If ψ is a conjunction of axioms of Q then

$$(1) \qquad Q \vdash \sim f(\phi) \supset : \psi \,.\, \sim f(\phi).$$

By the essential reflexivity of Q

$$(2) \qquad Q \vdash \psi \,.\, \sim f(\phi) : \supset fg(\text{“}(\psi \,.\, \sim f(\phi)) \textit{ is consistent}\text{”}).$$

Obviously $P \vdash \text{“}(\psi \,.\, \sim f(\phi))$ *is consistent*$\text{”} \supset \sim \text{“}(\psi \supset f(\phi))$ *is provable*” and hence

$$(3) \quad Q \vdash fg(\text{“}(\psi \,.\, \sim f(\phi)) \textit{ is consistent}\text{”} \supset \sim fg(\text{“}(\psi \supset f(\phi)) \textit{ is provable}\text{”}).$$

Therefore, by (1), (2) and (3) we have

$$Q \vdash \sim f(\phi) \supset \sim fg(\text{“}(\psi \supset f(\phi)) \textit{ is provable}\text{”}) \text{ and hence}$$
$$Q/R \vdash \sim \phi \supset \sim g(\text{“}(\psi \supset f(\phi)) \textit{ is provable}\text{”}), \text{ i.e.,}$$
$$Q/R \vdash g(\text{“}(\psi \supset f(\phi)) \textit{ is provable}\text{”}) \supset \phi.$$

On the other hand, let $Q/R \vdash \phi$, i.e., $Q \vdash f(\phi)$. Therefore $f(\phi)$ is provable from a conjunction ψ of a finite number of the axioms of Q, and hence

$$P \vdash \text{“}(\psi \supset f(\phi)) \textit{ is provable.}\text{”} \text{ Thus}$$
$$(a) \qquad \vdash g(\text{“}(\psi \supset f(\phi)) \textit{ is provable}\text{”})$$
$$(b) \qquad \vdash g(\text{“}(\psi \supset f(\phi)) \textit{ is provable}\text{”}) \supset \phi, \text{ hence}$$
$$(a) \cup (b) \vdash \phi.$$

This theorem provides a set of axioms for the induced theory whenever a set of axioms of the inducing theory is given. If the given set of axioms of the inducing theory is recursive then we get a recursive set of axioms for the induced theory. The latter statement follows already, under much more general conditions, from the theorem of Craig [1]; but the set of axioms given here is somewhat simpler and has clearer mathematical content.

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[3] This form of schema (b) was suggested to the author by Robert L. Vaught.

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SOLUTION TO A PROBLEM OF ROSE AND ROSSER

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In a recent article by Rose and Rosser [1], the question is raised concerning the possibility of proving the following theorem using only the first three of Łukasiewicz' axioms for infinite-valued logic together with his rules of inference [2]:

$$CCQPCQR \equiv CCPQCPR. \tag{3.51}$$

The question is not only interesting in itself, but sheds some light on problems of independence relating to Łukasiewicz' axioms. For example, in another recent paper [3], C. A. Meredith establishes the dependence of Łukasiewicz' fourth axiom, using only the first three of Łukasiewicz' axioms together with Rose and Rosser's Theorem 3.51.

The purpose of this paper is to establish a negative answer to the Rose-Rosser question. This will be done in a way which will illustrate the use of many-valued logics [4] as instruments for deciding questions of independence, and from this, one will be able to see that in deciding a negative answer to the Rose-Rosser question, a logic with at least four truth-values is required. To this end, let APQ be defined as $CCPQQ$ and consider the following axiom schemes and rule of inference:

Axiom schemes:

A1. $CPCQP$.
A2. $CCPQCCQRCPR$.
A3. $CAPQAQP$.

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