ON THE INTERVAL TOPOLOGY OF AN l-GROUP

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1. **Introduction.** Let G be an l-group in the sense of Birkhoff [**1**; **2**]. We consider the well-known interval topology of G, which is obtained by taking the family of all closed intervals of G as a sub-base for the closed sets. Birkhoff [1, Problem 104, p. 233] raised the question whether an arbitrary partially ordered group is a topological group with respect to its interval topology. This question was answered in the negative by Northam [**4**], who gave an example of an l-group which is not a Hausdorff space in its interval topology (and hence not a topological group). The purpose of this note is to show that this behavior of an l-group is far from "pathological," but actually is characteristic of large classes of l-groups. Several theorems describing such classes of l-groups are obtained, all following as a consequence of a rather elementary lemma (Lemma 3 below).

2. **Preliminaries.** We shall employ the terminology of nets due to Kelley [**3**]. For brevity we shall write "G is IH" for the statement "G is a Hausdorff space in its interval topology." By a *closed interval* in G we shall mean any subset of the form $\{x \in G \mid a \leqq x \leqq b\}$, $\{x \in G \mid x \geqq a\}$, or $\{x \in G \mid x \leqq a\}$, where a and b are arbitrary elements of G.

LEMMA 1. *If there exists a net $\{f(n),\ n \in D\}$ in a partially ordered set G such that $f(n)$ is eventually in the complement of any closed interval of G, then G is not IH.*

PROOF. A base for the open sets of the interval topology of G consists of all subsets of the form $\bigcap \{K_i \mid i=1, 2, \cdots, k\}$, where each K_i is the complement of a closed interval. Thus any net f satisfying the above hypothesis is eventually in each open set of G. Hence f converges to each point of G and G is not IH.

Our terminology and notation for l-groups is that of Birkhoff [**1**; **2**]. Let G be any commutative l-group and $\mathfrak{M}$ its set of meet-irreducible l-ideals. For each $M \in \mathfrak{M}$ it is known that the l-quotient-group G/M is a simply ordered group. Furthermore, $\bigcap \{M \mid M \in \mathfrak{M}\}$ is empty for any commutative l-group G. Hence there exists an isomorphism of G onto an l-subgroup of the direct product of a set of simply ordered groups $\{G_i \mid i \in I\}$, where each G_i is the l-quotient-group of G by

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some meet-irreducible l-ideal M [2, Theorem 36]. We identify any element x of G with the corresponding element $(x_1, x_2, \cdots, x_i, \cdots)$ of the direct product $\prod\{G_i | i \in I\}$. The direct product $\prod\{G_i | i \in I\}$ is ordered "componentwise": i.e., if a and b are elements of this product, we define $a \leqq b$ if and only if $a_i \leqq b_i$ for all $i \in I$. The identity element of G will be denoted by 0, the identity of G_i by 0_i. The additive notation will be used.

The following result, which is a consequence of Theorems 23 and 27 of [2], is also known.

LEMMA 2. *If G is a commutative l-group and M is a maximal l-ideal of G, then G/M is an Archimedean simply ordered group.*

3. **Results.** Our theorems are a consequence of the following lemma.

LEMMA 3. *Let G be an l-subgroup of the direct product $\prod\{G_i | i \in I\}$ of arbitrary l-groups. Let r and s be any members of the index set I. Suppose that there exists a net $\{f(m), m \in D\}$ of elements of G satisfying*

(i) *for any $k \in G_r$, $f_r(m)$ is eventually greater than k, and*

(ii) *$f_s(m) \leqq 0_s$ for all $m \in D$.*

Then G is not IH.

PROOF. Let $\{b(n), n \in E\}$ be any net of elements of G with the property that, given any $j \in G_s$, $b_s(n)$ is eventually less than j. Note that for any $n \in E$, there exists an element m_n in D such that the rth component of $f(m_n) + b(n)$ is greater than 0_r (one merely chooses m_n so that $f_r(m_n) > -b_r(n)$). Define $g(n) = f(m_n) + b(n)$. Then g is a net on E to G such that (i) $g_s(n)$ is eventually less than any given $j \in G_s$, and (ii) $g_r(n) > 0_r$ for all $n \in E$. Now consider the directed set $D \times E$ consisting of all pairs (m, n) for $m \in D$, $n \in E$, directed as usual by defining $(m_1, n_1) \leqq (m_2, n_2)$ if and only if $m_1 \leqq m_2$ and $n_1 \leqq n_2$. (We are using the same symbol $\leqq$ for the order relations in D, E, and $D \times E$.) Define $h(m, n) = f(m) + g(n)$. Then h is a net on $D \times E$ to G which satisfies the hypothesis of Lemma 1. To see this, suppose that J_a is the closed interval $\{x \in G | x \leqq a\}$, where a is an arbitrary element of G. Then there exists $m_0 \in D$ such that $f_r(m) > a_r$ for all $m \geqq m_0$, and hence $h(m, n)$ is in the complement of J_a whenever $m \geqq m_0$. Likewise, given the closed interval $J'_a = \{x \in G | x \geqq a\}$, there exists $n_0 \in E$ such that $g_s(n) < a_s$ for all $n \geqq n_0$; and hence $h(m, n)$ is eventually in the complement of J'_a. Thus $h(m, n)$ is eventually in the complement of any closed interval, and by Lemma 1 G is not IH.

Note that Lemma 3 does not require that the factor groups G_i be simply ordered.

THEOREM 1. *Let G be a commutative l-group containing a maximal l-ideal M. If there exists an element b in G which is incomparable with* 0 *and such that $b \notin M$, then G is not IH.*

PROOF. By Lemma 2, G/M is an Archimedean simply ordered group. Thus, since M is meet-irreducible, G may be considered as an l-subgroup of a direct product $\prod\{G_i \mid i \in I\}$ of simply ordered groups, where for a certain index $r \in I$, G_r is Archimedean. Since $b \notin M$, we have $b_r \neq 0_r$. Assume that $b_r > 0_r$ (otherwise consider $-b$). Since b is incomparable with 0, for some $s \in I$ we must have $b_s < 0_s$. The sequence $\{nb \mid n = 1, 2, \cdots\}$ then satisfies the hypothesis of Lemma 3.

THEOREM 2. *Let G be an l-group such that $G = \prod\{G_i \mid i \in I\}$, where the index set I contains more than one member. Then G is not IH.*

PROOF. It is obviously possible to construct a net in G satisfying the hypothesis of Lemma 3.

THEOREM 3. *Let G be an l-subgroup of $\prod\{G_i \mid i \in I\}$, where each G_i is an Archimedean simply ordered group. Then G is IH if and only if G is simply ordered.*

PROOF. If b is an element of G which is incomparable with 0, then for some $r \in I$ we have $b_r > 0_r$, and for some $s \in I$ we have $b_s < 0_s$. The sequence $\{nb \mid n = 1, 2, \cdots\}$ satisfies the hypothesis of Lemma 3: hence G is not IH. The converse is clear.

Since any Archimedean simply ordered group is an l-subgroup of the additive group of the real numbers, Theorem 3 asserts that any l-group of real-valued functions which is not simply ordered is not IH. This result thus includes Northam's example (an l-group of continuous real-valued functions) as a special case.

It remains an open question whether Theorem 3 can be extended (at least for commutative l-groups) to the case where some or all of the factor groups G_i are non-Archimedean. In this case there may exist no net in G satisfying the conditions of Lemma 3, as the example in the next section shows.

We obtain still another application of Lemma 3. If a is a positive element of the l-group G, we say that a is *Archimedean* if and only if for any $x \in G$ there exists a positive integer n with $na \geqq x$. We then have

THEOREM 4. *Let G be a commutative l-group containing an element b which is incomparable with* 0 *and such that $|b|$ is Archimedean. Then G is not IH.*

PROOF. We consider G as an l-subgroup of $\prod\{G_i | i\in I\}$, where each G_i is simply ordered. For some $r\in I$, $s\in I$, we have $b_r>0_r$, $b_s<0_s$. Since $b_r=|b|_r$, Lemma 3 may be applied to the sequence $\{nb | n=1, 2, \cdots\}$.

4. **An example.** We give a simple example of a commutative l-group which is not simply ordered and in which there exists no net satisfying the hypothesis of Lemma 3. Let Z be the integers in the usual ordering, and let $H=Z\times Z$. We order H lexicographically by defining $(m_1, n_1)<(m_2, n_2)$ if and only if $m_1<m_2$ or, when $m_1=m_2$, if $n_1<n_2$. Note that H is non-Archimedean in this ordering. Now consider the l-group $H\times H$ with the usual (componentwise) direct product ordering. Define $G=\{((i, j), (m, n))\in H\times H | i=m\}$. G is an l-subgroup of $H\times H$. Let $F=\{((i, j), (m, n))\in G | i=0 \text{ and } m=0\}$. Note that any element of G which is not in F is comparable with the identity element of G. Thus any net in G which has one component eventually positive also has the other component eventually positive.

It should be noted, however, that G is not IH. For consider the sequence defined by $f(n)=((0, n), (0, -n))$, $n=1, 2, \cdots$. The reader may verify that any closed interval of G which contains the range of a subsequence of f also contains the range of the entire sequence. We conclude that if x is any member of the sequence, and J is a closed interval of G which does not contain x, then the sequence f is eventually in the complement of J. This means that f converges in the interval topology to every element x in its range. Hence G is not IH.

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