REMARKS ON A THEOREM OF M. ALTMAN

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Introduction. A mapping \( f : X \to X \) of a Banach space \( X \) into itself is said \([1]\) to be locally an \( \epsilon \)-mapping in the narrow sense, if for every \( x \in X \) there exist two positive numbers \( \eta_x \) and \( \epsilon_x \) such that the condition

\[
x', x'' \in S(x, \epsilon_x) \quad \text{and} \quad \| f(x') - f(x'') \| < \eta_x
\]

implies

\[
\| x' - x'' \| < \epsilon_x,
\]

where \( S(x, \epsilon_x) \) denotes the spherical region (in \( X \)) with centre \( x \) and radius \( \epsilon_x \).

In \([1]\) M. Altman proved that: If the mapping \( f(x) = x - F(x) \), \( x \in X \), where \( F(x) \) is a completely continuous transformation of the Banach space \( X \) with range in \( X \), is locally an \( \epsilon \)-mapping in the narrow sense, then the image \( f(X) \) of \( X \) is an open subset of the space \( X \).

Using this theorem he obtained in particular some conditions under which a linear mapping \( f : X \to X \) of \( X \) into \( X \) is a mapping onto \( X \). \([2]\)

In this paper, the definition of locally an \( \epsilon \)-mapping in the narrow sense is slightly modified and some theorems on mappings onto are proved for the so-called polynomial mappings. \([3]\) It turns out that for these mappings the assumption that the mapping is locally an \( \epsilon \)-mapping in the narrow sense in every point \( x \in X \) (made in Altman's theorems referred to above) may be replaced by a much weaker assumption.

I. Definition 1. A mapping \( f : X \to Y \) of a metric space \( X \) with metric \( \rho_1 \) into a metric space \( Y \) with metric \( \rho_2 \) will be called an \( \epsilon \)-mapping in the narrow sense at the point \( y \in f(X) \), if there exists a point \( x \in f^{-1}(y) \) and two positive numbers \( \eta_x \) and \( \epsilon_x \) such that

\[
x', x'' \in S(x, \epsilon_x) \quad \text{and} \quad \rho_2[f(x'), f(x'')] < \eta_x
\]

implies

\[
\rho_1(x', x'') < \epsilon_x,
\]
where \( S(x, \varepsilon) \) denotes the spherical region (in \( X \)) with centre \( x \) and radius \( \varepsilon \).

Note that if \( f: X \to X \) is locally an \( \varepsilon \)-mapping in the narrow sense, then it is also an \( \varepsilon \)-mapping in the narrow sense at every point \( y \in f(X) \)—but, as can be easily seen, not conversely.

**Definition 2.** A mapping \( f: X \to Y \) of a metric space \( X \) into a metric space \( Y \) is called open at the point \( y \in f(X) \) if there exists a spherical region \( S(y, r) \), (in \( Y \)), with centre \( y \) and radius \( r \) such that \( S(y, r) \subset f(X) \).

**Theorem 1.** If \( F: X \to X \) is a completely continuous operator of a Banach space \( X \) into itself and \( f(x) = x - F(x) \) is an \( \varepsilon \)-mapping in the narrow sense at the point \( y \in f(X) \), then \( f: X \to X \) is open at the point \( y \in f(X) \).

The proof of this theorem is identical with that of Theorem 1 in [1].

**Definition 3.** A mapping \( f: X \to Y \) of a metric space \( X \) into a metric space \( Y \) is called a polynomial mapping if the condition:

\[
\{x_i\}_{i=1}^{n, 2, \ldots} \text{ does not contain a Cauchy (fundamental) subsequence,}
\]
implies that

\[
\{f(x_i)\}_{i=1}^{n, 2, \ldots} \text{ also does not contain a Cauchy subsequence.}
\]

We give now some examples of polynomial mappings:

1. A mapping \( f: X \to Y \) is called an \( \varepsilon \)-mapping in the narrow sense\(^4\) if there exist two positive numbers \( \eta \) and \( \varepsilon \) such that the condition \( p_2[f(x'), f(x'')] < \eta \), \( x', x'' \in X \) implies \( p_1(x', x'') < \varepsilon \), where \( p_1 \) and \( p_2 \) denote the metric in \( X \) and \( Y \) respectively. We shall show that an \( \varepsilon \)-mapping \( f: X \to Y \) in the narrow sense of a finite dimensional Banach space \( X \) into a metric space \( Y \) is a polynomial mapping.

   Indeed, if \( \{f(x_i)\}_{i=1,2,\ldots} \) contains a Cauchy subsequence \( \{f(x_i')\}_{i=1,2,\ldots} \), then for any \( \eta \) there is \( p_2[f(x_i'), f(x_i'')] < \eta \) for \( n, m \) sufficiently large. Hence \( p_1(x_i', x_i'') = \|x_i' - x_i''\| < \varepsilon \) and therefore \( \{x_i'\}_{i=1,2,\ldots} \) is a bounded sequence in \( X \). Thus, \( X \) being a finite dimensional Banach space, the sequence \( \{x_i'\} \) is compact (conditionally) and therefore it contains a convergent subsequence \( \{x_i''\}_{i=1,2,\ldots} \) which is a Cauchy subsequence of \( \{x_i\}_{i=1,2,\ldots} \).

2. Every mapping \( f: X \to Y \) of a finite dimensional Banach space \( X \) into a finite dimensional Banach space \( Y \), such that if \( \|x_i\| \to \infty \) then also \( \|f(x_i)\| \to \infty \), is a polynomial mapping.\(^5\)

3. If there exists a constant \( k > 0 \) such that

\[^4\text{See [2].}\]

\[^5\text{See [6, p. 1398, (c)].}\]
$\rho_2[f(x'), f(x'')] \geq k\rho_1(x', x'')$

then $f: X \to Y$ is a polynomial mapping ($\rho_1$ and $\rho_2$ denoting the metrics in $X$ and $Y$ respectively).

**Theorem 2.** If $f: X \to Y$ is a polynomial mapping of a complete metric space $X$ into a connected metric space $Y$ which is open at every point $y \in f(X) - J$, where $J \neq f(X)$ and $J \subset f(X)$ is a set which does not disconnect the space $Y$, then $f(X) = Y$.

**Proof.** Since $J \neq f(X)$ and $f: X \to Y$ is open at every point $y$ of $f(X) - J$, there exists a spherical region $S(y, r)$ (in $Y$) which is contained in $f(X)$. Denote by $U$ the union of all sets, open in $Y$, which are contained in $f(X)$. Evidently $\text{Fr}(U) \subset \text{Fr}[f(X)]$. Now suppose that there exists a point $y_0 \in Y - f(X)$ and let $y_1$ be any point belonging to $U$. Since the set $J$ does not disconnect $Y$, there exists in $Y - J$ a connected set $C$ containing $y_0$ and $y_1$. But the set $\text{Fr}(U)$ disconnects the space $Y$ between $y_0$ and $y_1$; therefore, there exists a point $y_2 \in [\text{Fr}(U) - J] \cap C \subset \text{Fr}[f(X)] - J$. By Theorem 1 of [5], $f(X)$ is closed in $Y$. Hence $y_2 \in f(X)$. By assumption, $f: X \to Y$ is open at every point $y \in f(X) - J$ in contradiction with the fact that

$$y_2 \in \text{Fr}[f(X)] - J \subset f(X) - J,$$

because $f: X \to Y$ is evidently not open at every point belonging to $\text{Fr}[f(X)]$. Thus the assumption that there exists a point $y_0 \in Y - f(X)$ leads to a contradiction. From Theorems 1 and 2 we obtain the following

**Theorem 3.** If $F: X \to X$ is a completely continuous operator of a Banach space $X$ into itself and $f(x) = x - F(x)$ is a polynomial mapping which is also an $\varepsilon$-mapping in the narrow sense at every point

$$y \in f(X) - J,$$

where $J \neq f(X)$ and $J \subset f(X)$ is a set which does not disconnect $X$, then $f(X) = X$.

Indeed, by Theorem 1, $f: X \to X$ is open at every point $y \in f(X) - J$ and hence by Theorem 2 used for $X = Y$, there is $f(X) = X$.

**Remark 1.** The function $f(x) = \arctan x$ ($F(x) = x - \arctan x$) de-

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6 Some generalizations of this theorem to general topological spaces will be given in [6].

7 $\text{Fr}(A)$ denotes the boundary of $A$ in $Y$, i.e., $\text{Fr}(A) = \overline{A} \cap \overline{Y - A}$.

8 See [3, p. 247]; also [4, p. 80].

9 See [5, p. 158].
fined on the real line $X$ is an example of a function which is locally an $e$-mapping in the narrow sense, but which is not a polynomial mapping. In this case we have $f(X) \neq X$.

**Remark 2.** From Theorem 3 it follows that if $F$ satisfies the assumptions of Theorem 3, then there exists a point $x$ such that $F(x) = x$ (i.e., $F: X \rightarrow X$ has a fixed point).

### References


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**A CHAINABLE CONTINUUM NO TWO OF WHOSE NONDEGENERATE SUBCONTINUA ARE HOMEOMORPHIC**

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R. D. Anderson and Gustave Choquet [1] gave an example of a plane continuum no two of whose nondegenerate subcontinua are homeomorphic. The object of this note is to point out that there is a chainable continuum having this property. The only change we make in the construction given in [1] is to replace the $n$-ods used by R. D. Anderson and Gustave Choquet by chainable continua $C_n$.

A subcontinuum $Y$ of a continuum $X$ is a separating continuum of $X$ if $X - Y$ is not connected and $\overline{X - Y} = X$. A subcontinuum $Y$ of a continuum $X$ is a strong separating continuum if:

1. $Y$ is a separating continuum of $X$,
2. $X - Y$ has two components, $X_1$ and $X_2$,
3. there are points $y_1, y_2 \in Y$ such that $y_i \notin \overline{X_i}$.

Let $V = \{ (x, y)/y = |x| \text{ and } -1 < x < 1 \}$. Let $C_n$ be formed from $n$ copies of $V$ and $n + 1$ "lines" so that each $V$ is a strong separating continuum of $C_n$ as in Figure 1 (for $n = 2$).

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