A MAXIMAL THEOREM[1]

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Let X denote the unit circle and L^p, $1<p<\infty$, the usual Lebesgue space. Given $f\in L^p$ there is a harmonic function u in the unit disc with L^p boundary value f. Set $f^*(x)=\sup_{r<1}|u(r,x)|$. The Hardy-Littlewood Maximal Theorem[2] asserts that there exists a constant B_p such that $\|f^*\|_p\leqq B_p\|f\|_p$. A similar theorem is given in higher dimensions by H. E. Rauch [2] and K. T. Smith [3] where X is now the unit sphere in n-space. These results are obtained by first proving a maximal ergodic theorem and then passing over to the maximal theorem. The purpose of this note is to remark that the maximal theorem is a trivial deduction from a maximal ergodic theorem which is itself completely standard, so that, in effect, there is very little to prove.

Before presenting the general procedure, I give an example which illustrates everything. Let X be the real line and take $f\in L^p$. The harmonic function in the upper half plane with boundary values f is

$$h(t,x)=\int_{-\infty}^{\infty}Q(t,x-y)f(y)dy$$

where Q is the Poisson kernel, $Q(t,x)=\pi^{-1}t(t^2+x^2)^{-1}$, $t>0$. We set $f^*(x)=\sup_{t>0}|h(t,x)|$ and the relevant maximal theorem is $\|f^*\|_p\leqq B_p\|f\|_p$. The only fact we need about the Poisson kernel is that the convolution operators $Q(t)$ form a semi-group having the symbolic form $Q(t)=\exp(-t\Lambda^{1/2})$ where $\Lambda^{1/2}$ is the positive square root of $\Lambda=-d^2/dx^2$. Now $P(t)=\exp(-t\Lambda)$ is a formal expression for the Gaussian semi-group of convolution operators having the Weierstrass kernel, $P(t,x)=(1/2)\pi^{-1/2}t^{-1/2}\exp\{-(4t)^{-1}x^2\}$. Put

$$g(t,x)=\int_{-\infty}^{\infty}P(t,x-y)f(y)dy.$$

We evidently have the relation

$$h(s,x)=\int_{-\infty}^{\infty}\phi(s,t)g(t,x)dt$$

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[2] The original Hardy-Littlewood paper is in Acta Math. vol. 54 (1930) pp. 81–116. A convenient reference is [4, pp. 244–247]. Both this theorem and the generalization to spheres are listed as exercises in [1, Exercises 7 and 8, p. 718].

where ϕ is determined by the equation

$$\exp(-s\lambda^{1/2}) = \int_0^\infty \phi(s, t)\exp(-t\lambda)dt.$$

It is to be noted that ϕ is independent of the explicit nature of the kernels P and Q. One easily calculates that $t\phi(s, t) = \psi(s^{-2}t)$ where $\psi(u) = \pi^{-1/2}(4u)^{-1/2}\exp\{-(4u)^{-1}\}$. What is important is that we have

(i) $$\int_0^t |\phi(s, t)|\, dt \leqq I,$$

(ii) the total variation in t of $t\phi(s, t)$ is not greater than V,

where I and V are constants independent of s. Now one has

$$|h(s, x)| \leqq \int_0^\infty |\phi(s, t)|\, |g(t, x)|\, dt.$$

Assume that ϕ, or some majorant of $|\phi|$, satisfies (i) and (ii), and set

$$a(t, x) = t^{-1}\int_0^\infty |g(u, x)|\, du,$$

and put $\bar{f}(x) = \sup_{t>0} a(t, x)$. It is evident that

$$|h(s, x)| \leqq \int_0^\infty |\phi(s, t)|\, d_t ta(t, x) = \int_0^\infty a(t, x)|\phi(s, t)|\, dt$$
$$+ \int_0^\infty |t\phi(s, t)|\, d_t a(t, x) \leqq (I + 2V)\bar{f}(x).$$

Thus one concludes that $\|f^*\|_p \leqq (I+2V)\|\bar{f}\|_p$. Next, we observe that $\bar{f}$ is simply the supremum of the averages of $|f|$ with respect to a probability semi-group corresponding to a measure-preserving flow on X, in this case the flow is Brownian motion. Hence the maximal ergodic theorem[3] is applicable; it says that for $1 < p < \infty$, $\|\bar{f}\|_p \leqq A_p\|f\|_p$. The maximal theorem, $\|f^*\|_p \leqq B_p\|f\|_p$, follows with $B_p = (I+2V)A_p$.

The specific deduction made thus far has little merit; the standard derivation of the Hardy-Littlewood Maximal Theorem, of [4, p. 245] uses the same reasoning except that the underlying flow is uniform

[3] Cf. [1, Chapter VIII, especially Theorem 7, p. 693]. Our indebtedness to the ideas there (which appeared earlier in J. Rat. Mech. Anal. vol. 5 (1956) pp. 129–178) is evident.

translation rather than Brownian motion. However, it is clear that the argument persists in a quite general context, and we intend to put this generality to use.

Take for X any set, $\mathfrak{F}$ a Borel field of subsets of X, and μ a completely additive measure defined on $\mathfrak{F}$. By a measure-preserving flow on X we mean the assignment to each $t>0$ and $E\in\mathfrak{F}$ of a subset E_t of X, measurable with respect to the canonical extension of μ, such that the measure of the symmetric difference of E_{t+h} and E_t tends to zero as h tends to zero from above. Let L^p be the Lebesgue space with respect to the measure μ; we confine our attention to the range $1<p<\infty$. The flow induces a strongly continuous semi-group of operators $\{P(t)\}$ or L^p. Given $f\in L^p$ we form $g(t, x)=P(t)f(x)$ and $\bar{f}(x) =\sup_{t>0} t^{-1}\int_0^t|g(u, x)|du$. The maximal ergodic theorem states that there exist constants A_p such that $\|\bar{f}\|p\leqq A_p\|f\|p$. The operators $P(t)$ can be considered to be defined for all L^p spaces simultaneously; for brevity we shall call $\{P(t)\}$ a probability semi-group. What we have proved above is

THEOREM. *Let $\{P(t)\}$ be a probability semi-group defined on a measure space X and suppose $\{Q(s)\}$, $s>0$, is a one-parameter family of operators subordinate to $\{P(t)\}$, i.e., $Q(s)=\int_0^\infty\phi(s, t)P(t)dt$. Let L^p be the Lebesgue space with respect to an invariant measure, and for $f\in L^p$ set $h(s, x)=Q(s)f(x)$, $f^*(x)=\sup_{s>0}|h(s, x)|$. If the subordinator $\phi(s, t)$, or some majorant of $|\phi|$, satisfies* (i) *and* (ii) *above then for $1<p<\infty$ there exists a constant B_p such that $\|f^*\|_p\leqq B_p\|f\|_p$.*

For an application consider the unit sphere X in n-dimensional space and the L^p spaces with respect to the uniform measure. Given $f\in L^p$ we let $u(r, x)$ be the function harmonic in $r<1$ with boundary values $f(x)$. Thus $\Delta u=0$ where Δ is the Laplacian. We may write $\Delta=-r^{-n}(\partial/\partial r)(r^n(\partial/\partial r))+r^{-2}\Lambda$ where Λ is the Beltrami operator on the unit sphere. It follows that $r\partial u/\partial r=\{(\Lambda+c^2)^{1/2}-c\}u$ where $n=2c+1$. Set $h(s, x)=u(\exp(-s), x)$; then $h(s, x)=Q(s)f(x)$ where $Q(s)$ has the symbolic form $Q(s)=\exp\{-s[(\Lambda+c^2)^{1/2}-c]\}$. Let $\{P(t)\}$ be the semi-group $P(t)=\exp(-t\Lambda)$. This is a probability semi-group corresponding to Brownian motion on the sphere which is a measure-preserving flow with respect to the uniform measure. $Q(s)=\int_0^\infty\phi(s, t)P(t)dt$ where the subordinator ϕ is determined by $\exp\{-s[(\lambda+c^2)^{1/2}-c]\}=\int_0^\infty\phi(s, t)\exp(-t\lambda)dt$. Using the calculation given above we find $t\phi(s, t)=\exp(cs-c^2t)\psi(s^{-2}t)$ whence it follows that (i) and (ii) are satisfied. The result is

COROLLARY. *Suppose $f\in L^p$ on the unit sphere and $u(r, x)$ is the* *function harmonic in the unit ball with boundary values $f(x)$. Then if $1<p<\infty$ there exists a constant B_p such that $\|f^*\|_p \leqq B_p\|f\|_p$ where $f^*(x) = \sup_{r<1} |u(r, x)|$.*

We turn now to the question of the validity of the maximal theorem for rather general probability semi-groups. Suppose $\{P(t)\}$ is a probability semi-group with symbolic form $P(t)=\exp(-t\Lambda)$. If χ is a function alternating of order ∞, i.e., a completely monotone mapping, on $(0, \infty)$ and $\chi(0)=0$ then $\{Q(t)\}$ is again a probability semi-group where $Q(s)=\exp\{-s\chi(\Lambda)\}$. A rigorous discussion of the infinitesimal generators Λ and $\chi(\Lambda)$ is irrelevant here since what is meant is that $Q(s)=\int_0^\infty \phi(s, t)P(t)dt$ where ϕ is determined by $\exp\{-s\chi(\lambda)\} = \int_0^\infty \phi(s, t)\exp(-t\lambda)dt$. (In general we should write $d_t\Phi(s, t)$ for $\phi(s, t)dt$, where for each s, $\Phi(s, t)$ is a function increasing from 0 to 1 on $[0, \infty]$.) Such a process of subordination takes probability semi-groups into probability semi-groups. We can assert the maximal theorem for the semi-group $\{Q(t)\}$ if ϕ satisfies (i) and (ii) above. (i) is trivial since $\phi(s, t)\geqq 0$ and $\int_0^\infty \phi(s, t)dt=1$. It remains to decide for what χ's (ii) holds. We shall now show that for $\chi(x)=x^c$, $0<c<1$, (ii) is valid. Here $\int_0^\infty \phi(s, t)\exp(-t\lambda)dt=\exp(-s\lambda^c)$ so that $t\phi(s, t) =\psi_c(s^{-1/c}t)$ where ψ_c is determined by

$$\int_0^\infty \psi_c(u)\exp(-u\lambda)du = c\lambda^{c-1}\exp(-\lambda^c).$$

(The ψ used above corresponds to $c=1/2$.) Inversion of Laplace transforms gives

$$\psi_c(u) = \pi^{-1}\int_0^\infty \exp\{x^{1/c}u\cos\theta - x\cos\theta c\} \cdot\sin\{x^{1/c}u\sin\theta - x\sin\theta c + \theta c\}dx$$

where θ may be chosen at will in the range $\pi/2\leqq\theta\leqq\pi$. Changing variables we calculate the derivative as

$$\psi_c'(u) = c\pi^{-1}\int_0^\infty \exp\{yu\cos\theta - y^c\cos\theta c\} \cdot\sin\{yu\sin\theta - y^c\sin\theta c + \theta(c+1)\}y^c dy.$$

Taking $\theta=\pi$ we have the estimate for $u>2$:

$$|\psi_c'(u)| \leqq c\pi^{-1}\int_1^\infty \exp\{-(1/2)\, yu\}y^c dy = O(u^{-1-c}).$$

Taking $\theta=\pi/2$, we obtain the uniform bound

$$|\psi_c'(u)| \leqq c\pi^{-1}\int_0^\infty \exp\{-y^c\cos(1/2)\pi c\}y^c dy.$$

Therefore $\int_0^\infty |\psi_c'(u)|du = V < \infty$. The total variation of $t\phi(s, t)$ is V, and so (ii) holds.

The reasoning of the last paragraph establishes

THEOREM. *Suppose there is a measure-preserving flow on a measure space X inducing probability semi-groups $\{Q(t)\}$, $t>0$, on the Lebesgue spaces L^p with respect to an invariant measure. If the family of operators $\{Q(t)\}$ is subordinate to a probability semi-group $\{P(t)\}$ via the formal relation $Q(t)=\exp(-t\Lambda^c)$, $P(t)=\exp(-t\Lambda)$, where $0<c<1$, then given $f\in L^p$ and $f^*(x)=\sup_{t>0}|Q(t)f(x)|$ we have for $1<p<\infty$, $\|f^*\|_p \leqq B_p\|f\|_p$ where the bounds, B_p, depend only on p and c.*

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