A MAXIMAL THEOREM

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Let $X$ denote the unit circle and $L^p$, $1 < p < \infty$, the usual Lebesgue space. Given $f \in L^p$ there is a harmonic function $u$ in the unit disc with $L^p$ boundary value $f$. Set $f^*(x) = \sup_{r < 1} |u(r, x)|$. The Hardy-Littlewood Maximal Theorem asserts that there exists a constant $B_p$ such that $\|f^*\|_p \leq B_p \|f\|_p$. A similar theorem is given in higher dimensions by H. E. Rauch [2] and K. T. Smith [3] where $X$ is now the unit sphere in $n$-space. These results are obtained by first proving a maximal ergodic theorem and then passing over to the maximal theorem. The purpose of this note is to remark that the maximal theorem is a trivial deduction from a maximal ergodic theorem which is itself completely standard, so that, in effect, there is very little to prove.

Before presenting the general procedure, I give an example which illustrates everything. Let $X$ be the real line and take $f \in L^p$. The harmonic function in the upper half plane with boundary values $f$ is

$$h(t, x) = \int_{-\infty}^{\infty} Q(t, x - y)f(y)dy$$

where $Q$ is the Poisson kernel, $Q(t, x) = \pi^{-1/2}(t^2 + x^2)^{-1}, t > 0$. We set $f^*(x) = \sup_{t > 0} |h(t, x)|$ and the relevant maximal theorem is $\|f^*\|_p \leq B_p \|f\|_p$. The only fact we need about the Poisson kernel is that the convolution operators $Q(t)$ form a semi-group having the symbolic form $Q(t) = \exp(-t\Lambda^{1/2})$ where $\Lambda^{1/2}$ is the positive square root of $\Lambda = -d^2/dx^2$. Now $P(t) = \exp(-t\Lambda)$ is a formal expression for the Gaussian semi-group of convolution operators having the Weierstrass kernel, $P(t, x) = (1/2)\pi^{-1/2}t^{-1/2} \exp\left\{-\frac{4t}{\pi}x^2\right\}$. Put

$$g(t, x) = \int_{-\infty}^{\infty} P(t, x - y)f(y)dy.$$ 

We evidently have the relation

$$h(s, x) = \int_{-\infty}^{\infty} \phi(s, t)g(t, x)dt$$

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2 The original Hardy-Littlewood paper is in Acta Math. vol. 54 (1930) pp. 81-116. A convenient reference is [4, pp. 244-247]. Both this theorem and the generalization to spheres are listed as exercises in [1, Exercises 7 and 8, p. 718].
where \( \phi \) is determined by the equation
\[
\exp(-s\lambda^{1/2}) = \int_0^\infty \phi(s, t) \exp(-t\lambda) dt.
\]

It is to be noted that \( \phi \) is independent of the explicit nature of the kernels \( P \) and \( Q \). One easily calculates that \( t\phi(s, t) = \psi(s^{-2}t) \) where \( \psi(u) = \pi^{-1/2}(4u)^{-1/2} \exp\{-(4u)^{-1}\} \). What is important is that we have

(i) \[
\int_0^t |\phi(s, t)| dt \leq I,
\]

(ii) the total variation in \( t \) of \( t\phi(s, t) \) is not greater than \( V \), where \( I \) and \( V \) are constants independent of \( s \). Now one has
\[
|h(s, x)| \leq \int_0^\infty |\phi(s, t)| |g(t, x)| dt.
\]

Assume that \( \phi \), or some majorant of \( |\phi| \), satisfies (i) and (ii), and set
\[
a(t, x) = t^{-1} \int_0^\infty |g(u, x)| du,
\]
and put \( f(x) = \sup_{t>0} a(t, x) \). It is evident that
\[
|h(s, x)| \leq \int_0^\infty |\phi(s, t)| d_a(t, x) = \int_0^\infty a(t, x) |\phi(s, t)| dt
\]
\[
+ \int_0^\infty |t\phi(s, t)| d_a(t, x) \leq (I + 2V)f(x).
\]

Thus one concludes that \( \|f^*\|_p \leq (I + 2V)\|f\|_p \). Next, we observe that \( f \) is simply the supremum of the averages of \( |f| \) with respect to a probability semi-group corresponding to a measure-preserving flow on \( X \), in this case the flow is Brownian motion. Hence the maximal ergodic theorem\(^8\) is applicable; it says that for \( 1 < p < \infty \), \( \|f\|_p \leq A_p \|f\|_p \). The maximal theorem, \( \|f^*\|_p \leq B_p \|f\|_p \), follows with \( B_p = (I + 2V)A_p \).

The specific deduction made thus far has little merit; the standard derivation of the Hardy-Littlewood Maximal Theorem, of \([4, p. 245]\) uses the same reasoning except that the underlying flow is uniform

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\(^8\) Cf. \([1, Chapter VIII, especially Theorem 7, p. 693]\). Our indebtedness to the ideas there (which appeared earlier in J. Rat. Mech. Anal. vol. 5 (1956) pp. 129–178) is evident.
transformation rather than Brownian motion. However, it is clear that the argument persists in a quite general context, and we intend to put this generality to use.

Take for $X$ any set, $\mathcal{F}$ a Borel field of subsets of $X$, and $\mu$ a completely additive measure defined on $\mathcal{F}$. By a measure-preserving flow on $X$ we mean the assignment to each $t > 0$ and $E \in \mathcal{F}$ of a subset $E_t$ of $X$, measurable with respect to the canonical extension of $\mu$, such that the measure of the symmetric difference of $E_{t+h}$ and $E_t$ tends to zero as $h$ tends to zero from above. Let $L^p$ be the Lebesgue space with respect to the measure $\mu$; we confine our attention to the range $1 < p < \infty$. The flow induces a strongly continuous semi-group of operators $\{P(t)\}$ or $L^p$. Given $f \in L^p$ we form $g(t, x) = P(t)f(x)$ and $f(t) = \sup_{t > 0} t^{-1} \int_0^t |g(u, x)| \, du$. The maximal ergodic theorem states that there exist constants $A_p$ such that $\|f\|_p \leq A_p \|f\|_p$. The operators $P(t)$ can be considered to be defined for all $L^p$ spaces simultaneously; for brevity we shall call $\{P(t)\}$ a probability semi-group. What we have proved above is

**Theorem.** Let $\{P(t)\}$ be a probability semi-group defined on a measure space $X$ and suppose $\{Q(s)\}, s > 0$, is a one-parameter family of operators subordinate to $\{P(t)\}$, i.e., $Q(s) = \int_0^s \phi(s, t) P(t) \, dt$. Let $L^p$ be the Lebesgue space with respect to an invariant measure, and for $f \in L^p$ set $h(s, x) = Q(s)f(x), f^*(x) = \sup_{s > 0} |h(s, x)|$. If the subordinator $\phi(s, t)$, or some majorant of $|\phi|$, satisfies (i) and (ii) above then for $1 < p < \infty$ there exists a constant $B_p$ such that $\|f^*\|_p \leq B_p \|f\|_p$.

For an application consider the unit sphere $X$ in $n$-dimensional space and the $L^p$ spaces with respect to the uniform measure. Given $f \in L^p$ we let $u(r, x)$ be the function harmonic in $r < 1$ with boundary values $f(x)$. Thus $\Delta u = 0$ where $\Delta$ is the Laplacian. We may write $\Delta = -r^{-n}(\partial^2/\partial r^2)(r^n(\partial/\partial r)) + r^{-2}\Lambda$ where $\Lambda$ is the Beltrami operator on the unit sphere. It follows that $r \partial u/\partial r = \{((\Delta + c^2)^{1/2} - c\} u$. Set $h(s, x) = u(\exp(-s), x)$; then $h(s, x) = Q(s)f(x)$ where $Q(s)$ has the symbolic form $Q(s) = \exp\{s[\Delta + c^2]^{1/2} - c\}$. Let $\{P(t)\}$ be the semi-group $P(t) = \exp(-t\Lambda)$. This is a probability semi-group corresponding to Brownian motion on the sphere which is a measure-preserving flow with respect to the uniform measure. Given $f \in L^p$ we let $u(\exp(-s), x)$ be the function harmonic in $r < 1$ with boundary values $f(x)$. Thus $\Delta u = 0$ where $\Delta$ is the Laplacian. We may write $\Delta = -r^{-n}(\partial^2/\partial r^2)(r^n(\partial/\partial r)) + r^{-2}\Lambda$ where $\Lambda$ is the Beltrami operator on the unit sphere. It follows that $r \partial u/\partial r = \{((\Delta + c^2)^{1/2} - c\} u$. Set $h(s, x) = u(\exp(-s), x)$; then $h(s, x) = Q(s)f(x)$ where $Q(s)$ has the symbolic form $Q(s) = \exp\{s[\Delta + c^2]^{1/2} - c\}$. Let $\{P(t)\}$ be the semi-group $P(t) = \exp(-t\Lambda)$. This is a probability semi-group corresponding to Brownian motion on the sphere which is a measure-preserving flow with respect to the uniform measure. Given $Q(s) = \int_0^s \phi(s, t) P(t) \, dt$ where the subordinator $\phi$ is determined by $\exp\{s[\Delta + c^2]^{1/2} - c\} = \int_0^s \phi(s, t) \exp(-t\Lambda) \, dt$. Using the calculation given above we find $t \phi(s, t) = \exp(cs - c^2t)\psi(s^2t)$ whence it follows that (i) and (ii) are satisfied. The result is

**Corollary.** Suppose $f \in L^p$ on the unit sphere and $u(r, x)$ is the
function harmonic in the unit ball with boundary values \( f(x) \). Then if \( 1 < p < \infty \) there exists a constant \( B_p \) such that \( \|f^*\|_p \leq B_p \|f\|_p \) where \( f^*(x) = \sup_{r<1} |u(r, x)| \).

We turn now to the question of the validity of the maximal theorem for rather general probability semi-groups. Suppose \( \{P(t)\} \) is a probability semi-group with symbolic form \( P(t) = \exp(-t\Lambda) \). If \( \chi \) is a function alternating of order \( \infty \), i.e., a completely monotone mapping, on \((0, \infty)\) and \( \chi(0) = 0 \) then \( \{Q(t)\} \) is again a probability semi-group where \( Q(s) = \exp\{-s\chi(\Lambda)\} \). A rigorous discussion of the infinitesimal generators \( \Lambda \) and \( \chi(\Lambda) \) is irrelevant here since what is meant is that \( Q(s) = \int_0^s \phi(s, t)P(t)dt \) where \( \phi \) is determined by \( \exp\{-s\chi(\Lambda)\} = \int_0^s \phi(s, t)\exp(-t\Lambda)dt \). (In general we should write \( \int_0^s \phi(s, t)dt \) for \( \phi(s, t) \), where for each \( s \), \( \phi(s, t) \) is a function increasing from 0 to 1 on \([0, \infty]\).) Such a process of subordination takes probability semi-groups into probability semi-groups. We can assert the maximal theorem for the semi-group \( \{Q(t)\} \) if \( \phi \) satisfies (i) and (ii) above, (i) is trivial since \( \phi(s, t) \geq 0 \) and \( \int_0^s \phi(s, t)dt = 1 \). It remains to decide for what \( \chi \)'s (ii) holds. We shall now show that for \( \chi(x) = x^c, 0 < c < 1 \), (ii) is valid. Here \( \int_0^s \phi(s, t)\exp(-t\Lambda)dt = \exp(-s\lambda^c) \) so that \( t\phi(s, t) = \psi_c(s^{-1/\epsilon}) \) where \( \psi_c \) is determined by

\[
\int_0^\infty \psi_c(u) \exp(-u\lambda)du = c\lambda^{c-1} \exp(-\lambda^c).
\]

(The \( \psi \) used above corresponds to \( c = 1/2 \).) Inversion of Laplace transforms gives

\[
\psi_c(u) = \pi^{-1} \int_0^\infty \exp\{x^{1/\epsilon}u \cos \theta - x \cos \theta c\}
\cdot \sin\{x^{1/\epsilon}u \sin \theta - x \sin \theta c + \theta c\} dx
\]

where \( \theta \) may be chosen at will in the range \( \pi/2 \leq \theta \leq \pi \). Changing variables we calculate the derivative as

\[
\psi'_c(u) = c\pi^{-1} \int_0^\infty \exp\{yu \cos \theta - y^c \cos \theta c\}
\cdot \sin\{yu \sin \theta - y^c \sin \theta c + \theta(c + 1)\} y^c dy.
\]

Taking \( \theta = \pi \) we have the estimate for \( u > 2 \):

\[
|\psi'_c(u)| \leq c\pi^{-1} \int_1^\infty \exp\{- (1/2) yu\} y^c dy = O(u^{-1-c}).
\]

Taking \( \theta = \pi/2 \), we obtain the uniform bound
\[
|\psi_c'(u)| \leq c\pi^{-1} \int_0^\infty \exp\{-y^c \cos(1/2)\pi c\} y^c dy.
\]

Therefore \(\int_0^\infty |\psi_c'(u)| \, du = V < \infty\). The total variation of \(t\phi(s, t)\) is \(V\), and so (ii) holds.

The reasoning of the last paragraph establishes

**Theorem.** Suppose there is a measure-preserving flow on a measure space \(X\) inducing probability semi-groups \(\{Q(t)\}, t > 0\), on the Lebesgue spaces \(L^p\) with respect to an invariant measure. If the family of operators \(\{Q(t)\}\) is subordinate to a probability semi-group \(\{P(t)\}\) via the formal relation \(Q(t) = \exp(-t\Delta^c), P(t) = \exp(-t\Delta)\), where \(0 < c < 1\), then given \(f \in L^p\) and \(f^*(x) = \sup_{t > 0} |Q(t)f(x)|\) we have for \(1 < p < \infty\), \(\|f^*\|_p \leq B_p \|f\|_p\), where the bounds, \(B_p\), depend only on \(p\) and \(c\).

**Bibliography**


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