sists of the set of all n such that $E_d n \in T$. The Gödel number of E_d is, of course, d and the Gödel number of n (i.e. of a string of 1's of length n) is $2^n - 1$. Thus A consists of all n such that $d * (2^n - 1) \in T_0$. We add to (S_2):

Axiom. $P1 - 11$.

Production. $Px - y \to Px1 - yy$.

P represents the set of all ordered pairs (i, j) such that $2^i = j$.

Then we add:

Production. $Px - y1,\ Cd - y - z,\ T_0 z \to Qx$.

In this system (S), "Q" represents A.

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Princeton University

THE AXIOM FOR CONNECTED SETS

R. L. STANLEY

1. **Introduction.** The basic motivation for this study was a desire to find a genuinely unified postulational principle which incorporated both the Axiom of Choice and the "axiom for sets," which latter means an appropriate analogue of the *Aussonderungsaxiom* to provide for the existence of sets. The possibility of thus uniting these two axiomatic principles has become especially interesting since adding the Axiom of Choice has been shown to be not only safe,[1] but necessary as well.[2] In particular, it was further hoped and expected that such a principle, when found, could be expressed naturally as a membership-equivalence statement—that is, essentially of the form

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[1] Gödel showed [1] that for certain systems of set theory, adding the Axiom of Choice does not bring inconsistency.

[2] Mendelson showed [2] that the Axiom of Choice is independent of the other, usual axioms for set theory, hence is indispensable for developments which employ it essentially.

$\ulcorner \cdots (\exists\beta)(\alpha)(\alpha\in\beta . \equiv . \cdots) \urcorner$, as are the axioms for sets in various systems.

Such a conceptually unified principle was found, and is presented here. This principle contrasts interestingly with the usual, so-called "Zorn lemmas" in two marked ways. First, it is conceptually much simpler than the lemmas, since it incorporates only the very "beginning" set-concepts, whereas the lemmas usually must be stated in terms of ideas which are defined relatively quite a bit later. Second, this principle is obviously and importantly much stronger than any standard Zorn lemma. The usual lemma is equivalent to the Axiom of Choice, once given the context of a well-developed, "pre-Choice" set theory within which to demonstrate that equivalence. The principle at hand, however, yields not only the Axiom of Choice, but also the axiom for sets, thus comprehending within itself the main foundation-stone of pre-Choice set theory as well as the usual, post-Choice materials.

2. **The axiom.** In its classical, or naive form, before any restrictions are installed to prevent logical paradoxes, this axiom is:

ACS.[3] If β is different from α_1 and α_2, and is not free in ϕ_1 or ψ_{12},[4] then

$$\vdash \ulcorner (R\psi . S\psi . T\psi) \supset (\exists\beta)(\alpha_1)(\alpha_1\in\beta . \equiv . \phi_1 . (\alpha_2)(\psi_{12} . \alpha_1 \neq \alpha_2 . \supset . \alpha_2 \tilde{\in} \beta)) \urcorner .$$

$\ulcorner R\psi \urcorner$, $\ulcorner S\psi \urcorner$, and $\ulcorner T\psi \urcorner$ mean respectively that ψ is reflexive (for α-terms which satisfy condition ϕ), symmetric, and transitive; formally, these are abbreviations of $\ulcorner (\alpha_1)(\phi_1 \supset \psi_{11}) \urcorner$, $\ulcorner (\alpha_1)(\alpha_2)(\psi_{12} \supset \psi_{21}) \urcorner$, and $\ulcorner (\alpha_1)(\alpha_2)(\alpha_3)(\psi_{12} . \psi_{23} . \supset \psi_{13}) \urcorner$.

This amounts to asserting the following: given any two conditions ϕ_1 and ψ_{12}, where ψ_{12} is reflexive for any elements which satisfy ϕ_1,

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[3] '*ACS*,' of course, denotes this new postulational principle. '*AC*' and '*AS*' will be used to abbreviate 'Axiom of Choice' and 'axiom for sets,' respectively. 'Axiom of Choice' means here Zermelo's familiar, unrestricted principle, conceived in terms of sets, to the effect that, for any set S of nonempty, nonoverlapping, member sets, s_i, there is an Auswahlmenge, or "choice-set," A, whose membership is made up of one and only one member from each such s_i. AS was characterized in section one above.

[4] In this discussion, ϕ_i and ψ_{ij}, for example, are used to represent the results of validly substituting α_i for α_1 in ϕ, and α_i and α_j for α_1 and α_2 in ψ, respectively. 'Validly' means here that quantifications are to be varied alphabetically, where necessary, to preserve freedom of substituted terms. For example, '$(x)(y)(x=w . \supset . x=y)$' becomes '$(z)(y)(z=x . \supset . z=y)$', not '$(x)(y)(x=x . \supset . x=y)$', when '$x$' is substituted for '$w$.' Except where specific departures are thus indicated, notational conventions are the same as those of [3], with small Greek letters representing set-variables, large Greek letters standing for arbitrary formulae, corners acting as quasi-quotation marks, and $\ulcorner \vdash \phi \urcorner$ abbreviating $\ulcorner$the closure (under universal quantification) of ϕ is a theorem$\urcorner$.

and ψ_{12} is absolutely symmetric and transitive, then there exists (at least) some set β whose members individually satisfy the condition ϕ_1, and collectively are connected under the (relational) condition $\sim\psi_{12}$; β is also "full" or maximal in the sense that adding any more members would falsify one or both of these two circumstances.

Of course, to serve as a postulational principle in any system, ACS must be restricted appropriately to forestall contradictions. Depending on the system, ACS will take various forms.

Where the system restricts all meaningfulness to type-conformant formulae, as does *Principia mathematica*, ACS will be restricted throughout, both in ϕ and ψ jointly, to conform wholly to the canons of types. In that case, ACS will still be equivalent to $(AS\cdot AC)$, AS being type-restricted as usual for such a system, and AC being wholly type-conformant in the first place.

In Zermelo's system of set theory, the appropriate version is

$$\mathrm{ACS}_z.\vdash\ulcorner(\gamma)(R\psi.S\psi.T\psi.\supset(\exists\beta)(\alpha_1)(\alpha_1\in\beta.\equiv.\alpha_1\in\gamma.\phi_1 \\ .(\alpha_2)(\psi_{12}.\alpha_1\neq\alpha_2.\supset.\alpha_2\tilde{\in}\beta)))\urcorner.$$

The equivalence of this form with the conjunction of AS_z and AC is conditional upon two more of Zermelo's axioms, those of Summation (ASm) and for the Set of Subsets (ASb). Within the pre-set context of Basal Logic (that is, the propositional calculus, quantification theory, and theory of identity), the following implications can be proven:

$$(ASm.ACS_z)\supset(AS_z.AC);$$
$$(ASb.AS_z.AC)\supset ACS_z.$$

Consequently, in a somewhat over-stated, but rather more suggestive form:

$$(ASm.ASb)\supset(ACS_z\equiv.AS_z.AC).$$

In Quine's system NF, adding AC leads to contradiction,[5] so not only AS, but AC also must be restricted in this system. If one considered only the logical paradoxes, the natural restriction for ACS_{nf} would require simply that $\ulcorner(\phi_1.\psi_{12})\urcorner$ be stratified. Such a form would certainly still support the disproof of the unlimited AC, however, so further weakening, now in its relational content, appears necessary to prevent ACS_{nf} from yielding the strong AC also, and thereby inconsistency.

[5] Specker derived [4] the negation of the strong axiom of choice as a theorem of Quine's system *New foundations*.

3. $ACS \supset AS$.[6] Consider the instance of ACS in which ϕ_1 is the appropriate condition of AS, and ψ_{12} is $\ulcorner(\alpha_1 = \alpha_2)\urcorner$. Since identity is absolutely reflexive, symmetric, and transitive, ACS reduces directly to

$$\vdash \ulcorner(\exists\beta)(\alpha_1)(\alpha_1 \in \beta . \equiv . \phi_1 . (\alpha_2)(\alpha_1 = \alpha_2 . \alpha_1 \neq \alpha_2 . \supset . \alpha_2 \tilde{\in} \beta))\urcorner.$$

Due to its impossible antecedent, the conditional is true for all values of its variables, hence the whole right side of the biconditional becomes equivalent to its part, the formula ϕ_1. This instance of ACS thus collapses into

$$\vdash \ulcorner(\exists\beta)(\alpha_1)(\alpha_1 \in \beta . \equiv \phi_1)\urcorner,$$

which is AS.

4. $ACS \supset AC$. Consider the instance of ACS in which ϕ_1 is $\ulcorner(\exists s_1)(\alpha_1 \in s_1 . s_1 \in S)\urcorner$, and ψ_{12} is $\ulcorner(\exists s_1)(\alpha_{1,2} \in s_1 . s_1 \in S)\urcorner$, with '$S$' denoting the usual set of nonempty sets, in AC. Plainly ϕ_1 implies ψ_{11}, and furthermore ψ_{12} is symmetric and transitive on α_1 and α_2, given AC's usual hypothesis that members of S are mutually exclusive. With its antecedent thus satisfied, ACS reduces to

$$\vdash \ulcorner(\exists\beta)(\alpha_1)(\alpha_1 \in \beta . \equiv . \phi_1 . (\alpha_2)(\psi_{12} . \alpha_1 \neq \alpha_2 . \supset . \alpha_2 \tilde{\in} \beta))\urcorner.$$

This asserts the existence of some class β such that: (1) if any element α_1 belongs to β, then α_1 is a member of some member of S, but no other element belongs both to β and to any S-member to which α_1 belongs; and (2) if any α_1 belongs to some member of S, but every other element which belongs to an S-member which contains α_1 is excluded from β, then α_1 belongs to β. Again given that S-members are mutually exclusive, this β clearly is a set whose membership is built by taking one and only one element from each member of S, and such a β meets the requirements for the usual Auswahlmenge, whose existence is the consequent in AC. AC thus follows from the assumption of ACS.

5. $(AS . AC) \supset ACS$. If the hypothesis of ACS holds, then its for-

[6] These demonstrations, sketched quite informally, trace out only the main outlines of the completely rigorous, lengthy, formal proofs. The proofs are "classical" or "natural" in the sense that no restrictions against paradox are incorporated, but neither are any steps taken which employ or yield paradoxical results. Such basic models of axioms and proofs are useful preliminary outlines, to be restricted in any direction and supplemented as necessary to suit any particular system's needs. For example, one such instance system would be type-theoretic; since no steps here violate type-restrictions, one needs merely to preface in general that all proof-lines with unspecified formulae be type-conformant.

mula ψ_{12} is symmetric and transitive, and reflexive for all α_1 such that ϕ_1 is true. AS is the main guarantee that a set exists for every specified, predicative condition, and AC is a generalization which is true for all values of 'S'. Given AS and AC, then, one may assign to 'S' the instantial value $\ulcorner \hat{s}_1(\exists\alpha_1)(\phi_1 . s_1 = \hat{\alpha}_1\phi_1 \cap \hat{\alpha}_1\hat{\alpha}_2\psi_{12}``\iota\alpha_1) \urcorner$. So defined, S consists of members s_i, whose own members in turn are all the elements α_j for which ϕ_j is true, and which satisfy the relational condition ψ_{jk} with some α_k which similarly belongs to s_i and satisfies ϕ_k.

For any s_1 to belong to S, there must exist some such α_1 for which ϕ_1 is true. By the hypothesis of ACS, ϕ_1 implies ψ_{11}, so that α_1 itself belongs to this image-set, and therefore s_1 contains at least the member α_1, satisfying the nonemptiness requirement in the hypothesis of AC. Furthermore, since ψ_{12} is a symmetric and transitive condition, any s_1 will consist simply of all the α_i for which ϕ_i holds and which satisfy condition ψ_{ij} with respect to any member of s_1. If any s_1 and s_2 have a common member α_1, then, they alike consist of all further α_i (within $\hat{\alpha}_1\phi_1$) which bear ψ_{i1} to α_1, and hence they must be identical. Thus different members of S are mutually exclusive, satisfying the rest of the AC-hypothesis.

With the antecedent thus established, the consequent in AC now holds unconditionally, guaranteeing that an Auswahlmenge A exists, containing one and only one member from each of the s_i-sets, which are themselves the members of S. Now since ϕ holds while ψ_{12} fails for any α_1 and α_2 which belong to different s_i-sets, A must consist of all and only those elements α_1 such that ϕ_1 is true, and such that no other α_2 for which ψ_{12} holds belongs to A. These are precisely the conditions, however, which define the set β in ACS, so such a set A is a suitable, connected set β for ACS.

In sum, given AS and AC, the antecedent of ACS leads to the consequent of ACS, so that $(AS.AC) \supset ACS$.

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Washington State University