A PROOF OF A FUNCTIONAL EQUATION RELATED TO THE THEORY OF PARTITIONS

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Introduction. The purpose of this note is to give a simple alternative proof for the following functional equation:

\[ \sum_{l=0}^{\infty} \left\{ \lambda((l + \alpha)z - i\beta) + \lambda((l + 1 - \alpha)z + i\beta) \right\} + \pi\zeta(\alpha^2 - \alpha + 1/6) + \pi(z(a^2 - a + 1/6) + 2\pi \alpha (\alpha - 1/2) \beta - 1/2), \]

where \( \lambda(t) = -\log(1 - e^{-2\pi t}) \) (the principal value), \( 0 \leq \alpha \leq 1, 0 < \beta < 1 \) (or \( 0 < \alpha < 1, 0 \leq \beta \leq 1 \)), and \( \Re(z) > 0 \). This formula was first proved in [2], and its applications to the theory of partitions have also been investigated (see [2; 3]).

The method we employ in this paper is essentially that of Rademacher [4]. Although our method is still based on the Mellin transform technique and the theory of the Hurwitz zeta-functions, we have made a thorough revision of Rademacher's original method, and thus a very short and direct proof of the functional equation can be given.

It may be noted that if we define a function \( \Lambda(z, \alpha, \beta) \) by

\[ \Lambda(z, \alpha, \beta) = \sum_{l=0}^{\infty} \left\{ \lambda((l + \alpha)z - i\beta) + \lambda((l + 1 - \alpha)z + i\beta) \right\} + \pi\zeta(\alpha^2 - \alpha + 1/6) - \pi \alpha (\alpha - 1/2) \beta - 1/2), \]

then (1) is written in the form

\[ \Lambda(z, \alpha, \beta) = \Lambda(z^{-1}, 1 - \beta, \alpha). \]

On the other hand, it is clear from the definition that

\[ \Lambda(z, \alpha, \beta) = \Lambda(z, 1 - \alpha, 1 - \beta). \]

Proof of the functional equation. To prove (1), we first assume that \( 0 < \alpha < 1, 0 < \beta < 1, \Re(z) > 0 \). Using the expansion

\[ \lambda(z) = -\log(1 - e^{-2\pi z}) = \sum_{n=1}^{\infty} n^{-1} e^{-2\pi zn} \]

and applying the Mellin formula
\[ e^{-v} = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \gamma^{-s} ds \quad (\Re(\gamma) > 0) \]

with \( c = 3/2 \), it follows that

\[
\sum_{l=0}^{\infty} \lambda((l + \alpha)z - i\beta) = \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} e^{-2\pi((l+\alpha)z-i\beta)n}
\]

\[
= \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \frac{e^{2\pi i \beta n}}{2\pi i n} \int_{3/2-i\infty}^{3/2+i\infty} \Gamma(s) \left\{ 2\pi(l + \alpha)zn \right\}^{-s} ds
\]

\[
= \frac{1}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} \frac{\Gamma(s)}{(2\pi z)^s} \left( \sum_{l=0}^{\infty} \frac{1}{(l + \alpha)^s} \right) \left( \sum_{n=1}^{\infty} \frac{e^{2\pi i \beta n}}{n^{1+s}} \right) ds.
\]

Therefore

\[
\sum_{l=0}^{\infty} \lambda((l + \alpha)z - i\beta) = \frac{1}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} \frac{\Gamma(s)}{(2\pi z)^s} \zeta(s, \alpha) \zeta(1 + s) ds,
\]

where \( \zeta(s, \alpha) = \sum_{l=0}^{\infty} (l + \alpha)^{-s} \) is the Hurwitz zeta-function, and \( \zeta(\beta)(s) = \sum_{n=1}^{\infty} \frac{e^{2\pi i \beta n}}{n^{1+s}} \) (\( \Re(s) > 0 \)).

Now the function \( \zeta(s, \alpha) \) (0 < \( \alpha \) < 1) can be expressed in terms of Hurwitz zeta-functions, i.e.,

\[
\zeta_\alpha(s) = \Gamma(1 - s)(2\pi)^{s-1} \left\{ e^{\pi i(1-s)/2} \zeta(1 - s, a)
\right. \\
+ e^{\pi i(s-1)/2} \zeta(1 - s, 1 - a) \}
\]

This relation is an easy consequence of the famous formula:

\[
\zeta(s, \alpha) = \frac{2\Gamma(1 - s)}{(2\pi)^{1-s}} \left\{ \sin(\pi s/2) \sum_{n=1}^{\infty} \frac{\cos(2\pi an)}{n^{1-s}}
\right. \\
+ \cos(\pi s/2) \sum_{n=1}^{\infty} \frac{\sin(2\pi an)}{n^{1-s}} \} \quad (\Re(s) < 0).
\]

From (3) we see that \( \zeta_\alpha(s) \) is an integral function of \( s \) (cf. Apostol [1]).

Returning to (2), we obtain further that

\[
\sum_{l=0}^{\infty} \left\{ \lambda((l + \alpha)z - i\beta) + \lambda((l + 1 - \alpha)z + i\beta) \right\}
\]

\[
= \frac{1}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} z^{-s} F(s; \alpha, \beta) ds
\]

with
The function $F(s; \alpha, \beta)$ satisfies the equation

$$F(s; \alpha, \beta) = F(-s; 1 - \beta, \alpha),$$

since we have, by (5) and (3),

$$F(s; \alpha, \beta) = \Gamma(s) \Pi(-s) \left[ e^{-s\pi/2} \left\{ \zeta(s, \alpha) \zeta(-s, \beta) + \zeta(s, 1 - \alpha) \zeta(-s, 1 - \beta) \right\} + e^{s\pi/2} \left\{ \zeta(s, \alpha) \zeta(-s, 1 - \beta) + \zeta(s, 1 - \alpha) \zeta(-s, \beta) \right\} \right].$$

which is unchanged when $s, \alpha, \beta$ are replaced by $-s, 1 - \beta, \alpha$ respectively.

Now in the integral on the right of (4), we move the path of integration from $\Re(s) = 3/2$ to $\Re(s) = -3/2$, and obtain

$$\frac{1}{2\pi i} \int_{3/2 - i\infty}^{3/2 + i\infty} z^{-s} F(s; \alpha, \beta) \, ds$$

$$= \frac{1}{2\pi i} \int_{-3/2 - i\infty}^{-3/2 + i\infty} z^{-s} F(s; \alpha, \beta) \, ds + R_1 + R_0 + R_{-1},$$

where $R_1, R_0, R_{-1}$ are the corresponding residues of the integrand at its simple poles $s = 1, 0, -1$ respectively. The displacement of the path of integration may easily be justified by considering the order of magnitude of the integrand. Actually, we have, writing $\zeta(s) = t,$

$$e^{-(s/2 - \delta)^2} \quad \text{for} \quad |\arg z| \leq \pi/2 - \delta < \pi/2,$$

$$\Gamma(s) \Pi(-s) = -\pi (s \sin \pi s)^{-1} = O\left( |t|^{-1} e^{-\pi |t|} \right),$$

$$\zeta(s, \alpha) = O\left( |t|^{-\gamma} \right)$$

(see [5, p. 276]),

and hence, by (7),

$$z^{-s} F(s; \alpha, \beta) = O\left( |t|^{-2} e^{-\pi |t|} \right).$$

The values of the residues are calculated as follows:

$$R_1 = (2\pi z)^{-1} \left\{ \zeta(2) + \zeta_{1-\beta}(2) \right\}$$

$$= (2\pi z)^{-1} \sum_{n=1}^{\infty} n^{-2} (e^{2\pi i\beta n} + e^{-2\pi i\beta n})$$

$$= (\pi z)^{-1} \sum_{n=1}^{\infty} n^{-2} \cos(2\pi \beta n) = \pi z^{-1} (\beta^2 - \beta + 1/6),$$
\[ R_0 = \zeta(0, \alpha) \zeta(1) + \zeta(0, 1 - \alpha) \zeta(1) \]
\[ = (1/2 - \alpha) \sum_{n=1}^{\infty} n^{-1} (e^{2\pi i \beta n} - e^{-2\pi i \beta n}) \]
\[ = 2i(1/2 - \alpha) \sum_{n=1}^{\infty} n^{-1} \sin(2\pi \beta n) \]
\[ = 2\pi i(1/2 - \alpha)(1/2 - \beta), \]

and, using (6),
\[ R_{-1} = -\pi \alpha (\alpha^2 - \alpha + 1/6). \]

Inserting these values and the relation (6) into the right member of (8), changing the variable \( s \) to \(-s\) and using (4), it is found that the required equation (1) holds for \( 0 < \alpha < 1, \ 0 < \beta < 1 \), and \( \Re(\varepsilon) > 0 \). The validity of (1) for the end-points of the interval of \( \alpha \) or \( \beta \) may then be established by means of letting \( \alpha \to 0, \ 1 - 0 \) or \( \beta \to 0, \ 1 - 0 \) in (1). This completes the proof of the functional equation.

**Bibliography**


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