A PROOF OF A FUNCTIONAL EQUATION RELATED TO THE THEORY OF PARTITIONS

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Introduction. The purpose of this note is to give a simple alternative proof for the following functional equation:

\[
\sum_{l=0}^{\infty} \left\{ \lambda((l + \alpha)z - i\beta) + \lambda((l + 1 - \alpha)z + i\beta) \right\} + \pi \zeta(\alpha^2 - \alpha + 1/6)
\]

\[
= \sum_{l=0}^{\infty} \left\{ \lambda((l + \beta)z^{-1} + i\alpha) + \lambda((l + 1 - \beta)z^{-1} - i\alpha) \right\}
\]

\[
+ \pi \zeta^{-1}(\beta^2 - \beta + 1/6) + 2\pi i(\alpha - 1/2)(\beta - 1/2),
\]

where \( \lambda(t) = -\log(1 - e^{-2\pi t}) \) (the principal value), \( 0 \leq \alpha \leq 1, 0 < \beta < 1 \) (or \( 0 < \alpha < 1, 0 \leq \beta \leq 1 \)), and \( \Re(z) > 0 \). This formula was first proved in [2], and its applications to the theory of partitions have also been investigated (see [2; 3]).

The method we employ in this paper is essentially that of Rademacher [4]. Although our method is still based on the Mellin transform technique and the theory of the Hurwitz zeta-functions, we have made a thorough revision of Rademacher's original method, and thus a very short and direct proof of the functional equation can be given.

It may be noted that if we define a function \( \Lambda(z, \alpha, \beta) \) by

\[
\Lambda(z, \alpha, \beta) = \sum_{l=0}^{\infty} \left\{ \lambda((l + \alpha)z - i\beta) + \lambda((l + 1 - \alpha)z + i\beta) \right\}
\]

\[
+ \pi \zeta(\alpha^2 - \alpha + 1/6) - \pi i(\alpha - 1/2)(\beta - 1/2),
\]

then (1) is written in the form

\[
\Lambda(z, \alpha, \beta) = \Lambda(z^{-1}, 1 - \beta, \alpha).
\]

On the other hand, it is clear from the definition that

\[
\Lambda(z, \alpha, \beta) = \Lambda(z, 1 - \alpha, 1 - \beta).
\]

Proof of the functional equation. To prove (1), we first assume that \( 0 < \alpha < 1, 0 < \beta < 1, \Re(z) > 0 \). Using the expansion

\[
\lambda(x) = -\log(1 - e^{-2\pi x}) = \sum_{n=1}^{\infty} n^{-1} e^{-2\pi xn}
\]

and applying the Mellin formula

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\[ e^{-y} = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \Gamma(s) y^{-s} ds \quad (\Re(y) > 0) \]

with \( c = 3/2 \), it follows that

\[
\sum_{l=0}^{\infty} \lambda((l + \alpha)z - i\beta) = \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} e^{-2\pi((l+\alpha)z-i\beta)n} = \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \frac{e^{2\pi i \beta n}}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} \Gamma(s) \left\{ 2\pi(l + \alpha)zn \right\}^{-s} ds
\]

\[
= \frac{1}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} \frac{\Gamma(s)}{(2\pi s)^s} \left( \sum_{l=0}^{\infty} \frac{1}{(l + \alpha)^s} \right) \left( \sum_{n=1}^{\infty} \frac{e^{2\pi i \beta n}}{n^{1+s}} \right) ds.
\]

Therefore

\[ \sum_{l=0}^{\infty} \lambda((l + \alpha)z - i\beta) = \frac{1}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} \frac{\Gamma(s)}{(2\pi s)^s} \xi(s, \alpha) \xi(1 + s) ds, \]

where \( \xi(s, \alpha) = \sum_{l=0}^{\infty} (l+\alpha)^{-s} \) is the Hurwitz zeta-function, and \( \xi(1 + s) = \sum_{n=1}^{\infty} n^{-s} e^{2\pi i \beta n} \) \( (\Re(s) > 0) \).

Now the function \( \xi_\alpha(s) \) \((0 < \alpha < 1)\) can be expressed in terms of Hurwitz zeta-functions, i.e.,

\[ \xi_\alpha(s) = \Gamma(1 - s)(2\pi)^{s-1} \left\{ e^{\pi i (1-s)/2} \xi(1 - s, \alpha) + e^{\pi i (s-1)/2} \xi(1 - s, 1 - \alpha) \right\}. \]

This relation is an easy consequence of the famous formula:

\[
\xi(s, \alpha) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \left\{ \frac{\cos(\pi s/2)}{\sum_{n=1}^{\infty} \sin(2\pi an)} \right\} \cos(\pi s/2) \sum_{n=1}^{\infty} \frac{\sin(2\pi an)}{n^{1-s}} \quad (\Re(s) < 0).
\]

From (3) we see that \( \xi_\alpha(s) \) is an integral function of \( s \) (cf. Apostol [1]).

Returning to (2), we obtain further that

\[
\sum_{l=0}^{\infty} \left\{ \lambda((l + \alpha)z - i\beta) + \lambda((l + 1 - \alpha)z + i\beta) \right\}
= \frac{1}{2\pi i} \int_{3/2-i\infty}^{3/2+i\infty} z^{-s} F(s; \alpha, \beta) ds
\]

with
(5) \[ F(s; \alpha, \beta) = \Gamma(s)(2\pi)^{-\frac{1}{2}} \left\{ \zeta(s, \alpha) \zeta(s+1) + \zeta(s, 1-\alpha) \zeta(s+1) \right\}. \]

The function \( F(s; \alpha, \beta) \) satisfies the equation

\[ F(s; \alpha, \beta) = F(-s; 1-\beta, \alpha), \]

since we have, by (5) and (3),

\[ F(s; \alpha, \beta) \]

\[ = \Gamma(s) \Gamma(-s) \left[ e^{-i\pi s/2} \left( \zeta(s, \alpha) \zeta(-s, \beta) + \zeta(s, 1-\alpha) \zeta(-s, 1-\beta) \right) \right. \]

\[ + e^{i\pi s/2} \left\{ \zeta(s, \alpha) \zeta(-s, 1-\alpha) + \zeta(s, 1-\alpha) \zeta(-s, \beta) \right\} \]

which is unchanged when \( s, \alpha, \beta \) are replaced by \(-s, 1-\beta, \alpha\) respectively.

Now in the integral on the right of (4), we move the path of integration from \( \Re(s) = 3/2 \) to \( \Re(s) = -3/2 \), and obtain

\[ \frac{1}{2\pi i} \int_{3/2+\infty}^{3/2+\infty} z^{-s} F(s; \alpha, \beta) ds \]

\[ = \frac{1}{2\pi i} \int_{3/2-\infty}^{-3/2-\infty} z^{-s} F(s; \alpha, \beta) ds + R_1 + R_0 + R_{-1}, \]

where \( R_1, R_0, R_{-1} \) are the corresponding residues of the integrand at its simple poles \( s=1, 0, -1 \) respectively. The displacement of the path of integration may easily be justified by considering the order of magnitude of the integrand. Actually, we have, writing \( \Im(s) = t, \)

\[ z^{-s} = O(e^{(\pi/2 - \delta)|t|}) \quad \text{for} \quad |\arg z| \leq \pi/2 - \delta < \pi/2, \]

\[ \Gamma(s) \Gamma(-s) = -\pi(s \sin \pi s)^{-1} = O(|t|^{-1} e^{-\pi|t|}), \]

\[ \zeta(s, \alpha) = O(|t|^\sigma) \quad \text{(see [5, p. 276])}, \]

and hence, by (7),

\[ z^{-s} F(s; \alpha, \beta) = O(|t|^{2e^{-1}e^{-4\pi|t|}}). \]

The values of the residues are calculated as follows:

\[ R_1 = (2\pi z)^{-1} \left\{ \zeta_\beta(2) + \zeta_{1-\beta}(2) \right\} \]

\[ = (2\pi z)^{-1} \sum_{n=1}^\infty n^{-2} (e^{2\pi i \beta n} + e^{-2\pi i \beta n}) \]

\[ = (\pi z)^{-1} \sum_{n=1}^\infty n^{-2} \cos(2\pi \beta n) = \pi z^{-1}(\beta^2 - \beta + 1/6), \]
\[ R_0 = \xi(0, \alpha)\xi_\beta(1) + \xi(0, 1 - \alpha)\xi_{1-\beta}(1) \]
\[ = (1/2 - \alpha) \sum_{n=1}^{\infty} n^{-1}(e^{2\pi i \beta n} - e^{-2\pi i \beta n}) \]
\[ = 2i(1/2 - \alpha) \sum_{n=1}^{\infty} n^{-1} \sin(2\pi n) \]
\[ = 2\pi i(1/2 - \alpha)(1/2 - \beta), \]

and, using (6),
\[ R_{-1} = -\pi \varepsilon(\alpha^2 - \alpha + 1/6). \]

Inserting these values and the relation (6) into the right member of (8), changing the variable \( s \) to \( -s \) and using (4), it is found that the required equation (1) holds for \( 0 < \alpha < 1, 0 < \beta < 1, \) and \( \Re(\varepsilon) > 0. \) The validity of (1) for the end-points of the interval of \( \alpha \) or \( \beta \) may then be established by means of letting \( \alpha \rightarrow +0, \ 1 - 0 \) or \( \beta \rightarrow +0, \ 1 - 0 \) in (1). This completes the proof of the functional equation.

**Bibliography**


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