CARTWRIGHT'S THEOREM ON FUNCTIONS BOUNDED AT THE INTEGERS

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1. The theorem of Cartwright [4] that for functions of exponential type not exceeding \( k < \pi \) bounded at the integers

\[
\sup_{-\infty < x < \infty} |f(x)| \leq A(k)M, \quad M = \sup |f(n)|, \quad n = 0, \pm 1, \pm 2, \ldots
\]

has been followed by many estimates of the value of \( A(k) \) [1; 3; 5; 6]. We shall generally mean by \( A(k) \) its smallest possible value. It has been shown [3] that as \( k \) approaches \( \pi \), \( A(k) \) must tend to infinity like \( \log \{1/(\pi - k)\} \). For \( k \) near \( \pi \) the estimates of \( A(k) \) from above and below are in close agreement [5]. But for small \( k \), for example \( 0 < k < \pi/2 \) we have only [1; 3]

\[
1 \leq A(k) \leq 2 + \frac{\pi}{3(\pi - k)}.
\]

The lower estimate can be improved. The function \( f(x) = \sin(\pi x/N) \) is of type \( \pi/N \) and if \( N \) is an odd integer \( \sup|f(x)| = \cos(\pi/2N) \) while \( \sup|f(x)| = 1 \). At any rate for some small values of \( k \) we have

\[
A(k) \geq \sec(k/2).
\]

It is a very natural conjecture that \( A(k) \) tends to unity as \( k \) tends to zero.

We are able to establish this conjecture in the following way.

Given the existence of \( A(k) \) and an upper estimate \( A_0(k) \) it follows from Bernstein's theorem [2, p. 206] that

\[
|f'(x)| \leq kA_0(k)M, \quad |f''(x)| \leq k^2A_0(k)M, \ldots.
\]

Use of these inequalities in various ways leads to different estimates for \( A(k) \), the best being

\[
A(k) \leq (1 - k^2/8)^{-1}, \quad (0 < k < 2^{1/3}),
\]

\[
A(k) \leq 2/(3 - k), \quad (2 < k < 3).
\]

Comparison with (1) shows that for small \( k \) our estimate (3) is asymptotically correct. The upper and lower estimates are each \( 1 + k^2/8 + O(k^4) \). For \( k = \pi/2 \) we have \( A(\pi/2) \geq 2^{1/2} \), which is also ob-

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tained in [4]. The estimate (3) is numerically 1.446 which exceeds 2^{1/2} by less than 3 per cent.

2. If the inequality (2) is integrated between \( x \) and the nearest integer \( n \) we have

\[
|f(x) - f(n)| \leq |x - n| kA_0(k)M \leq kA_0(k)M/2
\]

and hence

\[
|f(x)| \leq \left\{ 1 + kA_0(k)/2 \right\} M.
\]

The new constant \( 1 + kA_0(k)/2 \) will be less than \( A_0(k) \) if \( k < 2 \) and

\[
A_0(k) > 1 + kA(k)/2 \quad \text{or} \quad A_0(k) > 2/(2 - k).
\]

We can infer that

\[
|f(x)| \leq 2M/(2 - k)
\]

by an iterative argument. Set

\[
A_1(k) = 1 + kA_0(k)/2, \quad A_{n+1}(k) = 1 + kA_n(k)/2.
\]

Evidently

\[
|f(x)| \leq A_n(k)M = \left\{ 2/(2 - k) + (k/2)^n[A_0(k) - 2/(2 - k)] \right\} M.
\]

Since \( n \) can be arbitrarily large,

\[
|f(x)| \leq 2M/(2 - k).
\]

3. This simple argument is sufficient to establish the conjecture that \( A(k) \) tends to unity as \( k \) tends zero.\(^1\) A slight improvement is obtained by using a variant of Bernstein's theorem [2, p. 214] namely that if \( f(z) \) is an entire function of exponential type \( k \) bounded on the real axis then for \( 0 < 2\delta < \pi/k \)

\[
|f(t + \delta) - f(t - \delta)| \leq 2 \sin (\delta k) \sup |f(x)|.
\]

If this inequality is used with \( x = t \pm \delta \) and \( t \mp \delta \) the nearest integer, in place of (5) then \( \delta \leq 1/4 \) and

\[
\sup |f(x)| \leq M + 2 \sin (k/4)A_0(k)M.
\]

Arguing as before we now infer that

\[
\sup |f(x)| \leq \left\{ 1 - 2 \sin (k/4) \right\}^{-1} \sup |f(x)|,
\]

this inequality being valid for \( 0 < k < 2\pi/3 \).

\(^1\) R. P. Boas, Jr. gives us another proof. Suppose \( \rho k < \pi, \rho \) is an integer; then \( f(x)^\rho \) is of type \( \rho k \) and \( |f(x)|^\rho \leq M^\rho \). So \( |f(x)|^\rho \leq A(\rho k)M^\rho, |f(x)| \leq A(\rho k)^{1/\rho}M. \)
4. Lagrange's interpolation formula [7]

\[ f(x) = \frac{x - b}{a - b} f(a) + \frac{x - a}{b - a} f(b) + \lambda \frac{(x - a)(x - b)}{2} f''(\xi), \quad |\lambda| \leq 1, \]

may also be used with \( a = n < x < b = n + 1 \).

Since \( |f''(x)| \leq k^2 A_0(k) M \) by repeated use of Bernstein's theorem, we have

\[ |f(x)| \leq M + (x - a)(b - x)k^2 A_0(k) M/2 \leq M + k^2 A_0(k) M/8. \]

This leads to the estimate

\[ \sup |f(x)| \leq (1 - k^2/8)^{-1} \sup |f(n)| \]

valid for \( k < 2^{3/2} \).

It may be noted that if \( k > 2 \) our use of Lagrange's interpolation is inferior to the more elementary inequality

\[ |f(x) - f(n)| \leq |x - n| k A_0(k) M \]

when \( |x - n| < 1/2 - 1/k \). If we use (7) in the intervals \( n \leq x \leq n + 1/2 - 1/k, n + 1 - (1/2 - 1/k) \leq x \leq n + 1 \) and (6) for \( a \leq x \leq b \) with \( a = n + 1/2 - 1/k \) and \( b = n + 1 - (1/2 - 1/k) \), we evidently obtain

\[ |f(x)| \leq M + (1/2 - 1/k)k A_0(k) M + k^{-2} k^2 A_0(k) M/2. \]

This leads to the inequality

\[ \sup |f(x)| \leq \frac{2}{3 - k} \sup |f(n)| \]

valid for \( 2 < k < 3 \).

References