A RELATION BETWEEN RIESZ AND RIEMANN
SUMMABILITY

H. BURKILL AND G. M. PETERSEN

The Cesàro and the Riemann methods of summation are both
intimately connected with the theories of Fourier and trigonometric
series. It is therefore natural that the relation between these methods
should have been investigated in some detail. For instance Verblun-
sky [2] has proved that, when $k$ is a positive integer, summability
$(C, k-\delta)$ implies summability $(R, k+1)$; and Kuttner [1] has
proved that, for $k=1, 2$, summability $(R, k)$ implies summability
$(C, k+\delta)$.

Riesz's typical means generalise Cesàro summability and there is a
 corresponding generalisation for Riemann summability. Since both
these methods are appropriate for dealing with almost periodic func-
tions, we wish to establish a connection between them.

The notation we shall use is the following.

Riesz's typical means. If $k>0$, $0=\lambda_0<\lambda_1<\lambda_2<\cdots$ and

$$\sum_{\lambda_n<\omega} \left(1 - \frac{\lambda_n}{\omega}\right) u_n \rightarrow s$$

as $\omega \rightarrow \infty$, then $\sum u_n$ is said to be summable $(R, \lambda_n, k)$ to $s$.

Riemann summability. If $k$ is a positive rational number with odd
denominator, $0<\lambda_1<\lambda_2<\cdots$ and

$$u_0 + \sum_{n=1}^{\infty} \left(\frac{\sin \lambda_n k}{\lambda_n h}\right) u_n \rightarrow s$$

as $h \rightarrow 0$, then $\sum u_n$ is said to be summable $(R, k, \lambda_n)$ to $s$.

The object of this note is to prove the following result.

THEOREM. Suppose $\lambda_0=0$, $0<\lambda_{n+1}-\lambda_n \leq \delta$ for all $n$ and $\sum_{n=0}^{\infty} u_n$
is summable $(R, \lambda_n, 1)$ to $t$. Then $\sum_{n=0}^{\infty} u_n$ is also summable $(R, k, \lambda_n)$
to $t$ when $k>2$ (and $k$ is a rational number with odd denominator).

Before proceeding with the proof we might mention that, although
our result is new even when $\{\lambda_n\} = \{n\}$, from the point of view of
applications it would be more desirable to proceed from a nonintegral
Riesz mean to Riemann summability of an integral order. However
this problem appears to be a very formidable one because of the diffi-
culty of dealing with nonintegral Riesz means.

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Proof of the theorem. For \( n \geq 1 \) we put
\[
l_n = \sum_{r=0}^{n} \left( 1 - \frac{\lambda_r}{\lambda_n} \right) u_r
\]
so that the hypothesis of the theorem implies that \( l_n \to t \) as \( n \to \infty \). We have
\[
\lambda_n l_n = \lambda_n s_n - \sum_{r=0}^{n} \lambda_r u_r,
\]
where \( s_n = u_0 + u_1 + \cdots + u_n \), and so
\[
\lambda_{n+1} l_{n+1} - \lambda_n l_n = \lambda_{n+1} s_{n+1} - \lambda_n s_n - \lambda_{n+1} u_{n+1} = (\lambda_{n+1} - \lambda_n) s_n.
\]
It follows that, for \( n \geq 2 \),
\[
(1) \quad u_n = \frac{\lambda_{n+1} l_{n+1} - \lambda_n l_n}{\lambda_{n+1} - \lambda_n} - \frac{\lambda_{n+1} - \lambda_n l_{n-1}}{\lambda_n - \lambda_{n-1}};
\]
and, since \( \lambda_0 = 0 \), (1) is easily seen to hold for \( n = 1 \) also, whatever value is assigned to \( t_0 \).

Taking a suitable \( k > 2 \) we define \( \tau_h \), for \( h > 0 \), by
\[
\tau_h = u_0 + \sum_{n=1}^{\infty} \left( \frac{\sin \lambda_n h}{\lambda_n h} \right)^k u_n,
\]
the series being convergent since \( \lambda_{n+1} - \lambda_n \geq \rho \) and so \( u_n = o(\lambda_n) \). We put
\[
c(x) = \begin{cases} 1, & \text{for } x = 0, \\ \left( \frac{\sin x}{x} \right)^k, & \text{for } x \neq 0. \end{cases}
\]
Then, by (1) and since also \( u_0 = t_1 \),
\[
\tau_h = c(\lambda_0 h) t_1 + \sum_{n=1}^{\infty} c(\lambda_n h) \left( \frac{\lambda_{n+1} l_{n+1} - \lambda_n l_n}{\lambda_{n+1} - \lambda_n} - \frac{\lambda_{n+1} - \lambda_n l_{n-1}}{\lambda_n - \lambda_{n-1}} \right).
\]
Now \( t_n \) is bounded, \( c(\lambda_n h) = O(1/n^k) \) and \( \lambda_n/(\lambda_{r+1} - \lambda_r) = O(n) \). Hence the series for \( \tau_h \) may be written
\[
\tau_h = \sum_{n=1}^{\infty} \left( \frac{c(\lambda_{n+1} h) - c(\lambda_n h)}{\lambda_{n+1} - \lambda_n} - \frac{c(\lambda_n h) - c(\lambda_{n-1} h)}{\lambda_n - \lambda_{n-1}} \right) \lambda_n l_n \quad = \sum_{n=1}^{\infty} \alpha_{h, n} l_n,
\]
We wish to prove that this transformation is regular, i.e. that $\tau_h \to t$ as $h \to 0$. This is so if (and only if) the three Toeplitz conditions are satisfied, namely

(i) $\alpha_{h,n} \to 0$ as $h \to 0$, for all $n$,
(ii) $\sum_{n=1}^{\infty} |\alpha_{h,n}|$ is uniformly bounded for all $h > 0$,
(iii) $\sum_{n=1}^{\infty} \alpha_{h,n} \to 1$ as $h \to 0$.

The first and the third condition are easily shown to be satisfied. In the first place the continuity of $c(x)$ shows that $\alpha_{h,n} \to 0$ as $h \to 0$. Also

$$\sum_{n=1}^{\infty} \alpha_{h,n} = 1$$

for all $h > 0$. This is either proved directly or by taking $u_0 = 1$, $u_n = 0$ for $n \geq 1$ so that $\tau_n = 1$ for all $n$ and $\tau_h = 1$ for all $h > 0$.

We still have to prove (ii). We have

$$\alpha_{h,n} = \frac{(c(\lambda_{n+1} h) - c(\lambda_n h)) - (c(\lambda_n h) - c(\lambda_{n-1} h))}{\lambda_{n+1} h - \lambda_n h} \lambda_n h$$

$$= \{ c'(\eta_n) - c'(\xi_n) \} \lambda_n h$$

$$= (\eta_n - \xi_n) c''(x_n) \lambda_n h,$$

where $\lambda_{n-1} h < \xi_n < \lambda_n h < \eta_n < \lambda_{n+1} h$ and $\xi_n < x_n < \eta_n$. Now

$$c''(x) = k(k - 1) \left( \frac{\sin x}{x} \right)^{k-2} \left( \frac{x \cos x - \sin x}{x^2} \right)^2$$

$$+ k \left( \frac{\sin x}{x} \right)^{k-1} \left( \frac{-x^2 \sin x - 2x \cos x + 2 \sin x}{x^3} \right)$$

and so

$$|c''(x)| < \begin{cases} A & \text{if } 0 < x < 1, \\ A/x^k & \text{if } x \geq 1, \end{cases}$$

where $A$ is a constant.

Let $N$ be the integer such that

$$\lambda_N h < 1 \leq \lambda_{N+1} h.$$

For $n \leq N$,

$$|\alpha_{h,n}| < (\eta_n - \xi_n) A \lambda_n h < 2qh A \lambda_n h < 2A q^2 n h^2$$

and so
\[
\sum_{n=1}^{N} |\alpha_{h,n}| < 2Aq^2h^2 \sum_{n=1}^{N} n = Aq^2h^2N(N + 1) < Bh^2N^2 \\
\leq Bh^2(\lambda_N/\phi)^2 < B/\phi^2 = K,
\]
say.

For \( n > N \),
\[
|\alpha_{h,n}| < (\eta_n - \xi_n) \frac{A}{x_n^k} \lambda_n h < \frac{2Aq^2nh^2}{(\lambda_{n-1}h)^k} \leq \frac{2Aq^2}{\left(\frac{1}{2} \phi\right)^k h^{k-2}n^{k-1}}
\]
and so
\[
\sum_{n=N+1}^{\infty} |\alpha_{h,n}| < \frac{2Aq^2}{\left(\frac{1}{2} \phi\right)^k h^{k-2}} \sum_{n=N+1}^{\infty} \frac{1}{n^{k-1}} < \frac{C}{h^{k-2}(N + 1)^{k-2}}
\]
\[
< \frac{Cq^{k-2}}{(h\lambda_{N+1})^{k-2}} \leq Cq^{k-2} = L,
\]
say. Hence, for all \( h > 0 \),
\[
\sum_{n=1}^{\infty} |\alpha_{h,n}| < K + L
\]
and the theorem is therefore proved.

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REFERENCES


UNIVERSITY OF SHEFFIELD, ENGLAND AND
UNIVERSITY COLLEGE OF SWANSEA, WALES