REGULAR EXTENSIONS AND THE SOLVABILITY OF OPERATOR EQUATIONS

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Let $T_1$ be a closed linear operator in a complex Banach space. In this paper we are concerned with the effect of changes in the complex parameter $\lambda$ on the solvability of the equation $(T_1 - \lambda I)x = y$. By introducing the notion of a regular extension we are able to generalize a result of A. S. Markus [2].

In order to introduce extension terminology we consider $T_1$ to be an extension of a closed linear operator $T_0$, denoted $T_0 \subset T_1$. We use $D(T)$ to denote the domain of an operator $T$, $R(T)$ for the range of $T$ and $N(T)$ for the null space of $T$. $K(T)$ denotes the set of all elements $y$ such that the equation $T^*x = y$ is solvable for every positive integer $n$, a concept originally introduced by F. Riesz [3, p. 87].

We call a closed linear operator $T$ a regular extension at $\lambda$ if $T_0 \subset T \subset T_1$, $R(T - \lambda I) = R(T_1 - \lambda I)$ and $T - \lambda I$ has a bounded inverse. We call $T$ a regular extension near $\lambda_0$ if for every $\lambda$ in some neighborhood of $\lambda_0$, $T$ is a regular extension at $\lambda$. Let $\Gamma$ be any connected component of the open set consisting of all complex numbers $\lambda$ such that there exists a regular extension near $\lambda$.

**Theorem.** For each element $y$, either $y$ belongs to $K(T_1 - \lambda I)$ for all $\lambda$ in $\Gamma$, or there is no $\lambda$ in $\Gamma$ such that $y$ belongs to $K(T_1 - \lambda I)$ and the set of all $\lambda$ in $\Gamma$ such that $y$ belongs to $R(T_1 - \lambda I)$ has no accumulation point in $\Gamma$.

The above mentioned result of A. S. Markus [2] differs from this only in the hypothesis. He takes $\Gamma$ to be an open connected set such that for every $\lambda$ in $\Gamma$, $R(T_1 - \lambda I)$ is closed and $N(T_1 - \lambda I)$ is of finite dimension independent of $\lambda$. Our hypothesis implies that $R(T_1 - \lambda I)$ is closed for $\lambda$ in $\Gamma$, but admits certain instances of $N(T_1 - \lambda I)$ being infinite dimensional.

We find it convenient to employ several lemmas in the proof of the theorem. The idea for Lemma 1 came from a paper by I. C. Gokhberg.

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2 It is shown in the author's dissertation that if $R(T_1 - \lambda_0 I)$ is closed and $N(T_1 - \lambda_0 I)$ is finite dimensional then there exists a regular extension at $\lambda_0$. Such an extension is shown to be regular near $\lambda_0$ if and only if the dimension of $N(T_1 - \lambda I)$ is constant in some neighborhood of $\lambda_0$.  

415

**Lemma 1.** Let $T$ be a regular extension near $\lambda_0$ and $\{\lambda_j\}$ be a sequence such that $\lim_{j \to \infty} \lambda_j = \lambda_0$ and $\lambda_j \neq \lambda_0$ for each $j$. If $y$ is in $R(T_1 - \lambda_j I)$ for every $j$ then $y$ is in $K(T_1 - \lambda_0 I)$.

**Proof.** Because of linearity we may assume without loss of generality that $\lambda_0 = 0$. By the definitions of regular extensions there exists a neighborhood of $0$ such that for each $\lambda$ in this neighborhood $(T - \lambda I)^{-1}$ exists as a bounded operator defined on $R(T_1 - \lambda I)$. By possibly omitting a finite number of terms from $\{\lambda_j\}$ we may assume that $\{\lambda_j\}$ is contained in such a neighborhood. Thus $y$ belongs to $R(T_1 - \lambda_j I)$ for every $j$.

For each $j$ let $u_j$ be a solution of $(T - \lambda_j I)x = y$. Then $Tu_j = \lambda_j u_j + y$ and since $T^{-1}$ is bounded,

$$\|u_j\| \leq \|T^{-1}\| \cdot \|\lambda_j u_j + y\| \leq \|T^{-1}\| \cdot |\lambda_j| \cdot \|u_j\| + \|T^{-1}\| \cdot \|y\|.$$ 

Thus for sufficiently large $j$ we have

$$\|u_j\| \leq \frac{\|T^{-1}\| \cdot \|y\|}{1 - |\lambda_j| \cdot \|T^{-1}\|} \leq 2\|T^{-1}\| \cdot \|y\|,$$

and $\{u_j\}$ is a bounded sequence.

Now $\lim_{j \to \infty} Tu_j = y$ so $y$ belongs to $R(T)$ which is closed since it is the domain of the closed, bounded operator $T^{-1}$. Let $x_0$ be the solution of $Tx = y$. Notice that $(1/\lambda_j)(u_j - x_0) = T^{-1}u_j$ and because $\{\|T^{-1}u_j\|\}$ is bounded we have $\lim_{j \to \infty} u_j = x_0$. Assume the induction hypothesis that $x_0, x_1, \cdots, x_n$ satisfy the following conditions: $Tx_0 = y$, $Tx_k = x_{k-1}$ ($k = 1, 2, \cdots, n$) and $\lim_{j \to \infty} w_{j,n} = x_n$ where

$$w_{j,n} = \left(\frac{1}{\lambda_j}\right)^n \left[ u_j - \sum_{k=0}^{n-1} (\lambda_j)^k x_k \right].$$

Then $\lim_{j \to \infty} w_{j,n+1}$ exists since $\{w_{j,n}\}$ converges, $T^{-1}$ is bounded and $T w_{j,n+1} = w_{j,n}$. Let $x_{n+1} = \lim_{j \to \infty} w_{j,n+1}$ and observe $Tx_{n+1} = x_n$ because $T$ is closed. Thus by induction $y$ belongs to $K(T)$.

**Lemma 2.** If $T$ is a regular extension near $\lambda_0$ then $N(T_1 - \lambda_0 I) \subseteq K(T - \lambda_0 I)$.

**Proof.** Again assume $\lambda_0 = 0$ and $\{\lambda_j\}$ is a sequence such as in Lemma 1. Suppose $T_1 x = 0$. Then $(T_1 - \lambda_j I)(x/ -\lambda_j) = x$ and $x$ belongs to $K(T)$ by Lemma 1.
Lemma 3. If $T$ is a regular extension near $\lambda$, $m$ is a non-negative integer and $x \in D((T_1 - \lambda I)^{m+1})$ then $(T - \lambda I)^{-1}(T_1 - \lambda I)^{m+1}x \in R((T_1 - \lambda I)^m)$.

Proof. Let $\lambda = 0$, $S = T^{-1}$ and $x \in D(T_1^{m+1})$. Then $T_1x = T_1^{m+1}x - T_1ST_1^{m+1}x = 0$ so by Lemma 2, $x \in K(T_1)$ which is contained in $K(T)$. Let $v$ be such that $T_1v = x$ and observe $T_1(x - v) = T_1^mx - T_1^{m+1}x + ST_1^{m+1}x$.

Lemma 4. If $T$ is a regular extension near $\lambda$, $S = (T - \lambda I)^{-1}$ and $y \in K(T_1 - \lambda I)$ then for each non-negative integer $n$, $S^ny$ is defined and belongs to $K(T_1 - \lambda I)$.

Proof. Let $\lambda = 0$ and $y \in K(T_1)$. Assume the induction hypothesis that $S^ny \in K(T_1)$. Since $K(T_1)$ is contained in $R(T_1) = D(S)$ we see that $S^{n+1}y$ is defined. Consider the equation $T_1^mx = S^{n+1}y$. By the induction hypothesis there is a $u$ such that $T_1^{m+1}u = S^ny$ and then by Lemma 3 there is a $v$ such that $T_1^nv = S^{n+1}u$. Thus $S^{n+1}y = SS^ny = ST_1^{m+1}u = ST_1v$ and since $m$ is arbitrary we have $S^ny \in K(T_1)$.

Note that this shows $K(T - \lambda I) = K(T_1 - \lambda I)$.

Lemma 5. Let $T$ be a regular extension near $\lambda_0$ and $S = (T - \lambda_0 I)^{-1}$. Then $y \in K(T_1 - \lambda I)$ for $y$ in $K(T_1 - \lambda_0 I)$ and for all $\lambda$ such that $|\lambda - \lambda_0| \cdot ||S|| < 1$.

Proof. Let $\lambda_0 = 0$, $y \in K(T_1)$ and $\lambda$ satisfy $|\lambda| \cdot ||S|| < 1$. By Lemma 4, $S^ny$ is defined for all $n$ and $\sum_{j=0}^{n} (\lambda S)^j y$ is convergent because $||\lambda S|| < 1$. Since $S$ has a closed domain we see that $u = \sum_{j=0}^{\infty} (\lambda S)^j y \in D(S)$. Finally $x = Su$ satisfies

$$(T_1 - \lambda I)x = (T_1 - \lambda I)S \sum_{j=0}^{\infty} (\lambda S)^j y$$

$$= T_1Sy + T_1S \sum_{j=1}^{\infty} (\lambda S)^j y - \lambda S \sum_{j=0}^{\infty} (\lambda S)^j y$$

$$= y + \sum_{j=1}^{\infty} (\lambda S)^j y - \sum_{j=0}^{\infty} (\lambda S)^{j+1} y = y.$$

Lemma 6. $K(T_1 - \lambda I)$ is independent of $\lambda$ in $\Gamma$.

Proof. For each element $y$ let $\Gamma_y$ be the set of all $\lambda \in \Gamma$ such that $y \in K(T_1 - \lambda I)$. We will show that $\Gamma_y$ is both closed and open in $\Gamma$ and hence either void or all of $\Gamma$ since $\Gamma$ is connected.

If $\lambda_0 \in \Gamma \cap \Gamma_y$ then $\lambda_0 \in \Gamma_y$ by Lemma 1. Hence $\Gamma_y$ is closed in $\Gamma$. 

If $\lambda_0 \in \Gamma_y$ then Lemma 5 implies the existence of a neighborhood of $\lambda_0$ which is contained in $\Gamma$ and such that $y \in \mathcal{R}(T_1 - \lambda I)$ for every $\lambda$ in this neighborhood. For each such $\lambda$ we see that $\lambda \in \Gamma_y$ by Lemma 1. Thus $\Gamma_y$ is open in $\Gamma$.

With Lemma 6, the following observation completes the proof of the theorem.

Suppose $\lambda_0 \in \Gamma$ is an accumulation point of the set of all $\lambda$ such that $y \in \mathcal{R}(T_1 - \lambda I)$. By Lemma 1, $y \in \mathcal{K}(T_1 - \lambda_0 I)$ since $\mathcal{K}(T_1 - \lambda_0 I) = \mathcal{K}(T - \lambda_0 I)$ if $T$ is a regular extension near $\lambda_0$.

REFERENCES


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