In this paper we show that the ternary operation of a metric ternary distributive semi-lattice, a generalization of the ternary Boolean algebra of Grau [2], uniquely minimizes ternary distance. This generalizes a result of Birkhoff and Kiss [1, Corollary 1, p. 749]. We show, conversely, that in a metric space unique minimizing of ternary distance determines a ternary operation with respect to which the space is a ternary distributive semi-lattice. Particularly, a lattice whose graph satisfies the unique minimal ternary distance condition and certain finiteness conditions must be distributive. This answers a question proposed by Birkhoff and Kiss [1, p. 750].

1. Definitions and postulates. We state our results at the close of this section.

A ternary distributive semi-lattice, hereinafter abbreviated TDSL, is a set of 3 elements closed with respect to a ternary operation \((a, b, c)\) satisfying the following identities.

\[(T1) \ (a, a, b) = a.\]

\[(T2) \ (a, b, c) \text{ is invariant under all 6 permutations.}\]

\[(T3) \ (a, (b, c, d), e) = ((a, b, e), c, (a, d, e)).\]

Remark. The term, introduced by the author (Abstract 86, Bull. Amer. Math. Soc. vol. 54 (1948) p. 79), is a natural one in view of Lemma 3. If in Lemma 3 there exists \(a' \subseteq 3\) satisfying

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(T4) \((a, b, a') = b\) for all \(b \in \mathfrak{I}\)

then \(\mathcal{P}(a, \mathfrak{I})\) is a distributive lattice with \(a\) and \(a'\) as zero and unit elements. If also \(\mathfrak{I}\) satisfies:

(T5) For each \(a \in \mathfrak{I}\) there exists a complement \(a' \in \mathfrak{I}\) satisfying (T4), then \(\mathfrak{I}\) becomes the Ternary Boolean Algebra of Grau [2] and \(\mathcal{P}(a, \mathfrak{I})\) is a Boolean Algebra for each \(a \in \mathfrak{I}\).

By a suitable permutation of the letters in (T3) Sholander in [4, p. 801] was able to replace (T2) and (T3) by a single postulate (N). His (M) is (T1).

We remark here that by virtue of (T2), (T3) can be written and applied with many variations; particularly, the solo element in the right member can be \(b\) or \(d\).

In a metric space \(\mathcal{M}\) we denote distance by \(bc\) and introduce ternary distance \([x; b, c, d] = xb + xc + xd\).

We shall be concerned with an undirected graph \(\mathfrak{J}\) with no loops, i.e., the graph of a symmetric anti-reflexive binary relation \(R\) on a set of elements: \(aRa\) is false for all \(a \in \mathfrak{J}\) and \(aRb\) iff \(bRa\). Two elements \(b\) and \(c\) are vertices of an edge iff \(bRc\). Moreover, when \(\mathfrak{J}\) is connected, it is a metric space with respect to distance defined: \(bb = 0; bc = 1\) iff \(bRc; bc = n\) iff \(bRb_1R \cdots Rb_1 = c\) is a minimal such sequence. An even graph is one with no odd-sided polygons \(b_1Rb_2R \cdots Rb_{2n+1}Rb_1\).

The graph \(\mathfrak{J}(\mathfrak{P})\) of a partially ordered set \(\mathfrak{P}\) is defined by: \(bRc\) iff \(b < c\) or \(b > c\) (\(<:\) is covered by).

We shall deal with the following two minimal ternary distance postulates in a metric space \(\mathcal{M}\) and a corresponding ternary operation for each.

(U) For each (unordered) triple \(b, c, d \in \mathcal{M}\) there exists a unique \(t = [b, c, d] \in \mathcal{M}\) such that \([t; b, c, d] = (bc + cd + db)/2\).

(V) For each triple \(b, c, d \in \mathcal{M}\) there exists a unique \(s = [b, c, d] \in \mathcal{M}\) such that \([s; b, c, d] < [x; b, c, d]\) for all \(x \in \mathcal{M}, x \neq s\).

By virtue of Lemma 1 we shall see that (U) implies (V) in \(\mathcal{M}\).

Ternary betweenness relations and notation are defined as follows:

(TB) In a TDSL \(\mathfrak{I}\), \((bxc) \leftrightarrow (b, x, c) = x\).

(MB) In a metric space \(\mathcal{M}\), \(bxc \leftrightarrow bx + xc = bc\).

(VB) In a graph \(\mathfrak{J}\) satisfying (V) or (U), \([bxc] \leftrightarrow [b, x, c] = x\).

Finiteness conditions in terms of convex sets are defined as follows:

(TF) In a TDSL \(\mathfrak{I}\), \(\{x \in \mathfrak{I} | (bxc)\}\) is finite for all \(b, c \in \mathfrak{I}\).

(MF) In a metric space \(\mathcal{M}\), \(\{x \in \mathcal{M} | bxc\}\) is finite for all \(b, c \in \mathcal{M}\).

(VF) In a graph \(\mathfrak{J}\) satisfying (V) or (U), \(\{x \in \mathfrak{J} | [bxc]\}\) is finite for all \(b, c \in \mathfrak{J}\).

When (TF) holds we define the graph \(\mathfrak{J}(\mathfrak{I})\) of a TDSL \(\mathfrak{I}\) as follows:

\(bRc\) iff \(b \neq c\) and \((b, x, c) = b\) or \(c\) for all \(x \in \mathfrak{I}\). \(\mathfrak{J}(\mathfrak{I})\) will be connected,
as shown in Lemmas 4 and 7, and therefore metrizable in the manner described above.

We now summarize our results.

**Theorem 1.** If $\mathfrak{M}$ is simultaneously a TDSL and a metric space in which (TB) and (MB) are equivalent: $(bxc) \leftrightarrow bxc$, then (U) is satisfied (and also (V)).

**Theorem 2.** If a TDSL $\mathfrak{M}$ satisfies (TF), then the metric space $\mathfrak{M}(\mathfrak{M})$, as defined and metrized above, satisfies (U). Moreover (TB) and (MB) are equivalent.

**Theorem 3.** A metric space $\mathfrak{M}$ satisfying (U) is a TDSL with respect to the ternary operation $[b, c, d]$. Moreover (MB) is equivalent to (VB) (which is (TB)).

We define a unique ternary distance graph $\mathfrak{G}$, hereinafter called a UTD graph, as one satisfying (MF) and (V).

**Theorem 4.** A UTD graph satisfies (U) and is a TDSL with respect to the ternary operation $[b, c, d]$. Moreover (MB) and (VB) are equivalent.

**Theorem 5.** If every $a \in \mathfrak{L}$, a lattice with zero element $z$, is of finite dimension, and if the graph $\mathfrak{G}(\mathfrak{L})$ satisfies (MF) and (V), then $\mathfrak{L}$ is distributive.

2. **Ternary distributive semi-lattices.** In this section we consider a TDSL which is a metric space and prove Theorems 1 and 2.

**Lemma 1.** In any metric space $\mathfrak{M}$

(1) $[x; b, c, d] \geq (bc + cd + db)/2$,

(2) $[x; b, c, d] = (bc + cd + db)/2 \leftrightarrow bxc \cdot cxd \cdot dxb$.

**Proof.** (1) follows from taking one-half the sum of the inequalities $bx + xc \geq bc$, $cx + xd \geq cd$, $dx + xb \geq db$. Clearly equality holds simultaneously in all three iff equality holds in (1).

**Lemma 2.** In a TDSL $\mathfrak{M}$ $(btc) \cdot (ctd) \cdot (dtb)$ is satisfied uniquely by $t = (b, c, d)$, where $\cdot$ denotes logical conjunction.

This follows easily from (T1-2-3). See [3, 8.4 and 8.13].

**Proof of Theorem 1.** Since $(TB) \leftrightarrow (MB)$, by Lemma 2 we have $btc \cdot ctd \cdot dtb$ holding uniquely for $t = (b, c, d)$. Whence by (2) and (1) resp. $[x; b, c, d] = (bc + cd + db)/2 < [x; b, c, d]$ for all $x \neq t$.

**Lemma 3.** For each $a \in \mathfrak{A}$, a TDSL, the elements of $\mathfrak{A}$ constitute a distributive semi-lattice $\mathfrak{A}(a, \mathfrak{A})$, closed with respect to symmetric join of
meets of triples (called by Sholander a median semi-lattice) as follows:

1. The inclusion relation is given by $b \subseteq_c (a \sqcap c)$ (and $c \sqsupseteq b$) $\iff (a, b, c) = b$.
2. The zero element is $a$.
3. $\varphi(a, 3)$ is closed with respect to meet given by $b \sqcap_a c = (b, a, c)$.
4. Existence of common upper bound $b \subseteq_m c$ and $c \subseteq_m b$, implies the join exists and is given by $b \sqcup_a c = (b, m, c)$.
5. Distributivity: existence of $b \sqcup_a c$ implies $d \sqcap_a (b \sqcup_a c) = (d \sqcap_a b) \sqcup_a (d \sqcap_a c)$.
6. For all triples $b, c, d$ there exists $(b \sqcap_a c) \sqcup_a (c \sqcap_a d) \sqcup_a (d \sqcap_a b)$, which is $(b, c, d)$.

The proof is a routine application of the postulates and is done in [5, pp. 809–810].

**Lemma 4.** Every principal ideal of $\varphi(a, 3)$, namely $\varphi(a, m) \cap \{x \mid (axm)\}$, is a distributive lattice, which is finite if (TF) is satisfied.

**Proof.** The lemma follows from (4) of Lemma 3 and the fact that one distributive law implies the other.

**Lemma 5.** In a TDSL 3 $(abc) \cdot (acd) \iff (abd) \cdot (bcd)$. We prove this known result to illustrate applications of the postulates. If $(abc) \cdot (acd)$, then $(a, b, d) = ((a, a, b, c), d) = (a, b, c, d)$ $= (a, b, c) = b$ yielding $(abd)$. Also $(b, c, d) = ((a, b, c), c, d)$ $= ((a, c, d), b, (c, c, d)) = (c, b, c) = c$ so that $(bcd)$ subsists. The converse holds by symmetry.

**Lemma 6.** In $\varphi(a, 3)$, $b$ is covered by $c \neq b$: $b <_a c (c >_a b)$ iff $(a, b, c) = b$ and $(b, x, c) = b$ or $c$ for all $x \in 3$.

**Proof.** Let $b <_a c$. Then $(b, a, c) = (a, b, c) = b$ and $(abc)$. For arbitrary $x \in 3$ let $(b, x, c) = d$. Then also $(bdc)$ by Lemma 2. Applying Lemma 5 with roles of $c$ and $d$ interchanged, we obtain $(abd) \cdot (adc)$. By Lemma 3 $a \subseteq b \subseteq d \subseteq c$, and the hypothesis requires $d = b$ or $d = c$. Conversely, let $(b, x, c) = b$ or $c$ for all $x$ and $(b, a, c) = b$. Then immediately $b \subseteq c$. Assume $a \subseteq b \subseteq x \subseteq c$ so that $(abx) \cdot (axc) \cdot (abc)$. By Lemma 5 $(bxc)$ so that $x = (b, x, c)$, which must be $b$ or $c$ as desired.

**Lemma 7.** In a TDSL 3 satisfying (TF), $bRc$ in $\mathfrak{g}(3)$ iff $bR_\varphi c$ in $\mathfrak{g}(\varphi(a, 3))$, where $R_a$ is $\subseteq_a$. Thus $\mathfrak{g}(3)$ and $\mathfrak{g}(\varphi(a, 3))$ are isometric.

**Proof.** In $\mathfrak{g}(3) bRc$ iff $(b, x, c) = b$ or $c$ for all $x$ including $(b, a, c) = b$ or $c$. Hence by Lemma 6 $bRc$ iff $b \subseteq c$ in $\varphi(a, 3)$ iff $bR_\varphi c$ in $\mathfrak{g}(\varphi(a, 3))$.

**Lemma 8.** In a TDSL 3 satisfying (TF), (TB) and (MB) are equivalent.
Proof. Given \((abc)\). Then \((a, b, c) = b\) and \(b \subseteq \sigma(c)\). The latter is a finite distributive lattice by Lemma 4 and satisfies the Jordan-Dedekind chain condition. Therefore a chain \(a < a_1 < a_\ldots < a_m = b \in \sigma(a, c)\) exists and minimizes a sequence \(aR_a x_1 R_a \ldots R_a b\), where \(R_a = \leq a\). The corresponding sequence of \(\mathfrak{g}(3)\) of Lemma 7: \(aR_a x_1 \ldots R_a b = b\) is thus minimal so that \(ab = \delta_a[b]\), the dimension of \(b\) in \(\sigma(a, 3)\). Similarly \(ac = \delta_a[c]\). Again, \(b < a b_1 < a \ldots < a b_n = c\) minimizes sequences \(bR_a \ldots R_a c\) and by virtue of Lemma 7 yields a corresponding minimal sequence \(bR_b_1 R \ldots R_a c\) of \(g(3)\) of length \(bc = n\). The total chain \(a < a_1 < a \ldots a_m = b < a b_1 < a \ldots < a b_n = c\), again in view of the Jordan-Dedekind chain condition in \(\sigma(a, c)\), yields a minimal chain \(aR_a x_1 \ldots R_a c\) = \(bR_b_1 R \ldots R_a c\) = \(a\). Conversely suppose \(abc\) holds. Let \(d = (a, b, c)\). By Lemma 2 \((adb) \cdot (bdc) \cdot (cda)\). By the proof just completed \(adb - bdc - cda\). Hence \(0 = (ad + db - ab)/2 + (bd + dc - bc)/2 - (cd + da - ca)/2\) = \(bd - (ab + bc - ca)/2 = bd - (ac - ca)/2 = bd\). Thus \(b = d, (a, b, c) = b,\) and \((abc)\).

Proof of Theorem 2. Lemma 8 completes the hypothesis of Theorem 1.

3. Unique ternary distance graphs. We prove Theorems 3, 4 and 5 in this section.

Lemma 9. In a metric space satisfying \((U), (VB)\) and \((MB)\) are equivalent.

Proof. \([bcd]\) iff \([b, c, d]\) iff \([c; b, c, d] = bc + cc + cd = (bc + cd + db)/2\) iff \(bc + cd = bd\) iff \(bcd\).

Lemma 10. (Condition \((D)\) of Sholander [4, p. 804]). For each unordered triple \(b, c, d \in \mathfrak{m}\), a metric space satisfying \((U)\), there exists a unique \(s \in \mathfrak{m}\) such that \(bsc = csc = dsc\), namely \(s = [b, c, d]\).

Proof. By \((U)\) there exists unique \(s = [b, c, d]\) such that \([s; b, c, d] = (bc + cd + db)/2\). We apply Lemma 1. By \((MT2)\) \(bsc = csc = dsc\), and for \(x \neq s\) \([x; b, c, d] > (bc + cd + db)/2\) so that at least one of \(bxc, cxd, dxb\) fails.

Lemma 11. In any metric space \(abc \cdot acd < abd \cdot bcd\).

This is an elementary property of metric spaces.

Proof of Theorem 3. The metric betweenness relation \(bcd\) satisfies the set of conditions \(\Sigma_1(D, B_1, F)\) of Sholander [4, pp. 803–805]: \((D)\) by Lemma 10; \((B_1)\) \(aba \rightarrow a = b,\) trivially; and \((F)\) \(abc \cdot acd\)
By Lemma 9 the equivalent between
ness relation \([bcd]\) also satisfies \(\Sigma_1\). Sholander showed in [4, 4.10] that the corresponding ternary operation \([b, c, d]\) satisfies his conditions (M) and (N). The latter, he showed in [3, 8.3], are equivalent to (T1-2-3).

**Corollary to Theorem 3.** If a metric space \(\mathcal{M}\) satisfies (U) and for some pair \(a, a' \in \mathcal{M}\) \(axa'\) for all \(x \in \mathcal{M}\), then \(\mathcal{P}(a, 3) = \mathcal{P}(a, a')\) is a distributive lattice with \(a\) and \(a'\) as zero and unit elements.

**Proof.** By Theorem 3 \(\mathcal{M}\) becomes a TDSL \(\mathcal{F}\) under \([b, c, d]\) with \([axa']\) for all \(x \in \mathcal{F}\). I.e., \(a \leq x \leq a'\) for all \(x\). Lemma 4 completes the proof.

**Lemma 12.** A necessary and sufficient condition that a connected graph \(\mathcal{G}\) be even is that \(bRc\) implies \(bx - cx = \pm 1\). Furthermore, a UTD graph is even.

**Proof.** Given \(\mathcal{G}\) is even and suppose \(bRc\). Then \(1 = bc \geq bx - cx \geq -bc = -1\). But \(bx \neq cx\) since \(bx - cx + 1 \equiv bx + cx + bc \equiv 0 \pmod{2}\). Hence \(bx - cx = \pm 1\). Conversely suppose \(\mathcal{G}\) is not even. Two adjacent vertices \(b, c\) and the opposite vertex \(x\) of a smallest odd-sided polygon give \(bc = 1\) and \(bx = cx\). Moreover \([c; x, b, c] = cx + 1 = bx + 1 = [b; x, b, c]\) \(\leq xy + by + cy = [y; x, b, c]\) for \(b \neq y \neq c\). Hence \(b\) and \(c\) (and possibly \(y\) also) are tied for minimal ternary distance from \(x, b, c\) so that \(\mathcal{G}\) is not a UTD graph. We have thus proved the contrapositives of the converse and the second statement.

We may note at this point that in a UTD graph the ternary operation \([b, c, d]\) satisfies (T2) trivially by symmetry. It also satisfies (T1). For if \(x \neq a\), \([x; a, a, b] = ax + (ax + bx) > ax + ax + ab = [a; a, a, b]\) so that \(a = [a, a, b]\). We shall circumvent a direct proof of (T3), which would be tedious.

**Lemma 13.** In a UTD graph (MB) and (VB) are equivalent.

**Proof.** First suppose \(abc\). If \(x \neq b\), \([x; a, a, b] = ax + (ax + bx) > ac = ab + bc = [b; a, b, c]\). Hence \(b = [a, b, c]\) and \([abc]\) subsists. Conversely, suppose \([abc]\). We shall prove by induction on \(n = bc\) that \(abc\) follows. We note that \(abc\) holds trivially for \(n = 0\). When \(n = 1\), \(ab = ac = a + bc = ab + bc = [c; c, b, d]\) by the first part of this proof and leads to the contradiction \([a, b, c] = [c, x, b, c] = c \neq b = [a, b, c]\) and \([c; c, b, d] = ac + bc\) as desired. Assume \([a, b, c] = b\) implies \(abc\) whenever \(n \leq k\). Consider \([a, b, c] = b\) with \(n = bc = k + 1\). Let \(bRb_0\) with \(b_0\) on minimal \(b - c\) chain: \(bb_0 = 1\) and \(bb_0 + b_0c = bc\). Since \(b = [a, b, c]\), \(ab + bc = [b; a, b, c] < [b_0; a, b, c]\)
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Thus \( ab < ab_0 \) and \( ab_0 = ab + 1 \) by Lemma 12. Now \([b; a, b_0, c] = ab + 1 + bc > ab + bc = ab + 1 + b_0 c = ab + b_0 c = [b_0; a, b, c] \). Also for \( x \neq b \) we apply hypothesis and Lemma 12 to obtain
\[
[x; a, b_0, c] = [x; a, b, c] + (b_0 x - bx) \geq 1 + [b; a, b, c] + (\pm 1) \geq [b; a, b, c] = ab + bc = ab_0 + b_0 c = [b_0; a, b, c].
\]
Uniqueness of minimality in (V) requires that \( b_0 = [a, b_0, c] \) or \([a, b_0, c] \). But \( b_0 c = bc - 1 = k \).

By the induction hypothesis \( ab_0 c \) subsists. Hence \( ac = ab_0 + b_0 c = ab + bc \) yielding \( abc \). The induction is complete.

**Proof of Theorem 4.** Let \( s = [b, c, d] \). Then for \( x \neq s \)
\[
[s; b, s, c] = (bs + cs + ds) - ds < bx + cx + (dx - ds) \leq bx + cx + sx = [x; b, s, c].
\]
Thus \( s = [b, s, c] \) and \([b sc] \) subsists. Similarly \([csd] \) and \([dsb] \). Then \( bsc - c sd - dsb \) by Lemma 13. By Lemma 1 and (V) we have \([s; b, c, d] = (bc + cd + db) / 2 < [x; b, c, d] \) for all \( x \neq s \). This is \( U \). Hence Theorem 4 now follows from Theorem 3.

**Lemma 14.** In any lattice or semi-lattice if \( aRbRcRdRa \), where \( R \) is \( \leq \), alternate \( R \)'s are opposite directional covering.

This follows by definition of covering and uniqueness of join and meet when they exist.

**Lemma 15.** In a lattice \( \mathcal{L} \), for which \( \mathcal{G}(\mathcal{L}) \) is a UTD graph with respect to the ternary operation determined by the metric, \( b \leq c \) in \( \mathcal{L} \) iff \( b \neq c \) and \([b, x, c] = b \) or \( c \) for all \( x \in \mathcal{G}(\mathcal{L}) \).

**Proof.** If \( b \leq c \) in \( \mathcal{L} \), then by Lemma 12 \( \mathcal{G}(\mathcal{L}) \) is even and \( bx - cx = \pm 1 = \pm bc \) in \( \mathcal{G}(\mathcal{L}) \). Thus \([bcx] \) or \([cbx] \), i.e., \([b, x, c] = c \) or \( b \). If \( b \neq c \), then \( b = c \) or there exists \( r \in \mathcal{G}(\mathcal{L}) \) with \( b, r, c \) all distinct such that \([brc] \) or \([br, c] = r \neq b \) or \( c \).

**Proof of Theorem 5.** Finite dimensionality of the elements of \( \mathcal{L} \) makes \( \mathcal{G}(\mathcal{L}) \) well defined and connected through \( z \), so that it is a UTD graph. Hence by Theorems 3 and 4 \( \mathcal{G}(\mathcal{L}) \) is a TDSL with respect to the operation \([a, b, c] \), and (MB) is equivalent to (VB) (which is (TB)). Moreover all the lemmas are valid and applicable. By Lemma 3

\( \mathcal{P}(z, \mathcal{G}(\mathcal{L})) \) is a distributive semi-lattice with the same zero element \( z \) of \( \mathcal{L} \). We shall show that \( \mathcal{P}(z, \mathcal{G}(\mathcal{L})) \) is isomorphic to \( \mathcal{L} \) under the identity correspondence \( c \leftrightarrow c \). Combining the results of Lemmas 6 and 15 \( b \leq zc \) in \( \mathcal{P}(z, \mathcal{G}(\mathcal{L})) \) iff \( b \leq c \) in \( \mathcal{L} \). Accordingly, it will be sufficient to show that

(S) \( b < c \) in \( \mathcal{P}(z, \mathcal{G}(\mathcal{L})) \) implies \( b < c \) in \( \mathcal{L} \).

We employ an induction on \( n = zb \), the distance from \( z \) to \( c \) in \( \mathcal{G}(\mathcal{L}) \), and note (S) is trivially true for \( n = 1: z < c \) in \( \mathcal{P}(z, \mathcal{G}(\mathcal{L})) \) iff \( z < c \) in \( \mathcal{L} \).

Assume (S) holds for \( n \leq k \). Now consider \( b < c \) with \( zb = k \) and \( zc = k + 1 \) in \( \mathcal{G}(\mathcal{L}) \), and assume that \( b > c \) in \( \mathcal{L} \). From all necessarily

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finite descending chains \( b > c > \cdots \) in \( \mathcal{L} \) select one with an earliest agreement of \( > \) and \( \geq \): \( b = c_0 < c_1 < \cdots < c_{r-1} < c_r \leq c_{r+1} \geq \cdots \). By Lemma 4 the ideal \( \mathcal{P}(z, c_r) \) of \( \mathcal{P}(z, \mathcal{J}(\mathcal{L})) \) is a distributive lattice. Hence by lower semi-modularity \( c_{r-1} > c_r \geq c_{r+1} \). On the other hand \( c_{r-1} > c_r > c_{r+1} < d < c_{r-1} \), where the direction of the last two coverings are required by Lemma 14. If \( r = 1 \), the induction hypothesis requires \( b > d < c_2 \) contradicting \( c_2 < d < b \). If \( r > 1 \), then \( c_{r-1} \) contradicts the minimality of \( r \). Hence our assumption \( b > c \) is false, and the induction on \( n = zc \) for validity of (S) is complete. Therefore \( \mathcal{P}(z, \mathcal{J}(\mathcal{L})) \) is isomorphic to \( \mathcal{L} \) and is a distributive lattice (rather than a semi-lattice). Thus \( \mathcal{L} \) itself is distributive.

References


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