In this paper we show that the ternary operation of a metric ternary distributive semi-lattice, a generalization of the ternary Boolean algebra of Grau [2], uniquely minimizes ternary distance. This generalizes a result of Birkhoff and Kiss [1, Corollary 1, p. 749]. We show, conversely, that in a metric space unique minimizing of ternary distance determines a ternary operation with respect to which the space is a ternary distributive semi-lattice. Particularly, a lattice whose graph satisfies the unique minimal ternary distance condition and certain finiteness conditions must be distributive. This answers a question proposed by Birkhoff and Kiss [1, p. 750].

1. **Definitions and postulates.** We state our results at the close of this section.

A ternary distributive semi-lattice, hereinafter abbreviated TDSL, is a set of 3 elements closed with respect to a ternary operation \((a, b, c)\) satisfying the following identities.

\[(T1) \quad (a, a, b) = a.\]
\[(T2) \quad (a, b, c) \text{ is invariant under all } 6 \text{ permutations}.\]
\[(T3) \quad (a, (b, c, d), e) = ((a, b, e), c, (a, d, e)).\]

**Remark.** The term, introduced by the author (Abstract 86, Bull. Amer. Math. Soc. vol. 54 (1948) p. 79), is a natural one in view of Lemma 3. If in Lemma 3 there exists \(a' \in S\) satisfying
(T4) \((a, b, a') = b\) for all \(b \in \mathcal{S}\)
then \(\mathcal{S}(a, 3)\) is a distributive lattice with \(a\) and \(a'\) as zero and unit elements. If also \(3\) satisfies:

(T5) For each \(a \in 3\) there exists a complement \(a' \in 3\) satisfying (T4), then \(3\) becomes the Ternary Boolean Algebra of Grau [2] and \(\mathcal{S}(a, T)\) is a Boolean Algebra for each \(a \in 3\).

By a suitable permutation of the letters in (T3) Sholander in [4, p. 801] was able to replace (T2) and (T3) by a single postulate (N). His (M) is (T1).

We remark here that by virtue of (T2), (T3) can be written and applied with many variations; particularly, the solo element in the right member can be \(b\) or \(d\).

In a metric space \(\mathfrak{M}\) we denote distance by \(bc\) and introduce ternary distance \([x; b, c, d] = xb + xc + xd\).

We shall be concerned with an undirected graph \(\mathcal{G}\) with no loops, i.e., the graph of a symmetric anti-reflexive binary relation \(R\) on a set of elements: \(aRa\) is false for all \(a \in \mathcal{G}\) and \(aRb\) iff \(bRa\). Two elements \(b\) and \(c\) are vertices of an edge iff \(bRc\). Moreover, when \(\mathcal{G}\) is connected, it is a metric space with respect to distance defined: \(bb = 0\); \(bc = 1\) iff \(bRc\); \(bc = n\) iff \(bRb_1R \cdots Rb_n = c\) is a minimal such sequence. An even graph is one with no odd-sided polygons \(b_1Rb_2R \cdots Rb_{2n+1}Rb_1\).

The graph \(\mathcal{G}(\mathcal{S})\) of a partially ordered set \(\mathcal{S}\) is defined by: \(bRc\) iff \(b < c\) or \(b > c\) (< \(c\) is covered by).

We shall deal with the following two minimal ternary distance postulates in a metric space \(\mathfrak{M}\) and a corresponding ternary operation for each.

(U) For each (unordered) triple \(b, c, d \in \mathfrak{M}\) there exists a unique \(\tau = [b, c, d] \in \mathfrak{M}\) such that \([\tau; b, c, d] = (bc + cd + db)/2\).

(V) For each triple \(b, c, d \in \mathfrak{M}\) there exists a unique \(\tau = [b, c, d] \in \mathfrak{M}\) such that \([\tau; b, c, d] = [x; b, c, d]\) for all \(x \in \mathfrak{M}\) for \(\tau \neq x\).

By virtue of Lemma 1 we shall see that (U) implies (V) in \(\mathfrak{M}\).

Ternary betweenness relations and notation are defined as follows:

(TB) In a TDSL \(3\), \((bxc) \leftrightarrow (b, x, c) = x\).

(MB) In a metric space \(\mathfrak{M}\), \(bxc \leftrightarrow bx + xc = bc\).

(VB) In a graph \(\mathcal{G}\) satisfying (V) or (U), \([bxc] \leftrightarrow [b, x, c] = x\).

Finiteness conditions in terms of convex sets are defined as follows:

(TF) In a TDSL \(3\), \(\{x \in 3\mid (bxc)\}\) is finite for all \(b, c \in 3\).

(MF) In a metric space \(\mathfrak{M}\), \(\{x \in \mathfrak{M}\mid bxc\}\) is finite for all \(b, c \in \mathfrak{M}\).

(VF) In a graph \(\mathcal{G}\) satisfying (V) or (U), \(\{x \in \mathcal{G}\mid bxc\}\) is finite for all \(b, c \in \mathcal{G}\).

When (TF) holds we define the graph \(\mathcal{G}(3)\) of a TDSL \(3\) as follows:
\(bRc\) iff \(b \neq c\) and \((b, x, c) = b\) or \(c\) for all \(x \in 3\). \(\mathcal{G}(3)\) will be connected,
as shown in Lemmas 4 and 7, and therefore metrizable in the manner described above.

We now summarize our results.

**Theorem 1.** If \( S \) is simultaneously a TDSL and a metric space in which (TB) and (MB) are equivalent: \((bxc) \leftrightarrow bxc\), then (U) is satisfied (and also (V)).

**Theorem 2.** If a TDSL \( S \) satisfies (TF), then the metric space \( g(S) \), as defined and metrized above, satisfies (U). Moreover (TB) and (MB) are equivalent.

**Theorem 3.** A metric space \( M \) satisfying (U) is a TDSL with respect to the ternary operation \([b, c, d]\). Moreover (MB) is equivalent to (VB) (which is (TB)).

We define a unique ternary distance graph \( g \), hereinafter called a UTD graph, as one satisfying (MF) and (V).

**Theorem 4.** A UTD graph satisfies (U) and is a TDSL with respect to the ternary operation \([b, c, d]\). Moreover (MB) and (VB) are equivalent.

**Theorem 5.** If every \( a \in L \), a lattice with zero element \( z \), is of finite dimension, and if the graph \( g(L) \) satisfies (MF) and (V), then \( L \) is distributive.

2. **Ternary distributive semi-lattices.** In this section we consider a TDSL which is a metric space and prove Theorems 1 and 2.

**Lemma 1.** In any metric space \( M \)

\[(MT1) [x; b, c, d] \geq (bc+cd+db)/2,\]
\[(MT2) [x; b, c, d] = (bc+cd+db)/2 \leftrightarrow bxc \cdot cxd \cdot dbx.\]

**Proof.** (MT1) follows from taking one-half the sum of the inequalities \( bx+xc \geq bc \), \( cx+xd \geq cd \), \( dx+xb \geq db \). Clearly equality holds simultaneously in all three iff equality holds in (MT1).

**Lemma 2.** In a TDSL \( S \) \((btc) \cdot (ctd) \cdot (dtb)\) is satisfied uniquely by \( t = (b, c, d) \), where \( \cdot \) denotes logical conjunction.

This follows easily from (T1-2-3). See \([3, 8.4 \text{ and } 8.13]\).

**Proof of Theorem 1.** Since (TB) \( \leftrightarrow \) (MB), by Lemma 2 we have \( btc \cdot ctd \cdot dtb \) holding uniquely for \( t = (b, c, d) \). Whence by (MT2) and (MT1) resp. \([x; b, c, d] = (bc+cd+db)/2 < [x; b, c, d]\) for all \( x \neq t \).

**Lemma 3.** For each \( a \in S \), a TDSL, the elements of \( S \) constitute a distributive semi-lattice \( \varnothing (a, S) \), closed with respect to symmetric join of
meets of triples (called by Sholander a median semi-lattice) as follows:

1. The inclusion relation is given by \( b \subseteq_c c \) (and \( c \supseteq_b b \)) \( \iff \) \( (a, b, c) = b \).
2. The zero element is \( a \).
3. \( \mathcal{O}(a, 3) \) is closed with respect to meet given by \( b \cap_a c = (b, a, c) \).
4. Existence of common upper bound \( b \subseteq_a m \) and \( c \subseteq_a m \), implies the join exists and is given by \( b \cup_a c = (b, m, c) \).
5. Distributivity: existence of \( b \cup_a c \) implies \( d \cap_a (b \cup_a c) = (d \cap_a b) \cup_a (d \cap_a c) \).
6. For all triples \( b, c, d \) there exists \( b \cap_a c \cup_a (c \cap_a d) \cup_a (d \cap_a b) \), which is \( (b, c, d) \).

The proof is a routine application of the postulates and is done in [5, pp. 809-810].

**Lemma 4.** Every principal ideal of \( \mathcal{O}(a, 3) \), namely \( \mathcal{O}(a, m) = \{ x \mid (ax = m) \} \), is a distributive lattice, which is finite if (TF) is satisfied.

**Proof.** The lemma follows from (4) of Lemma 3 and the fact that one distributive law implies the other.

**Lemma 5.** In a TDSL 3 \( (abc) \cdot (acd) \iff (abd) \cdot (bcd) \).

We prove this known result to illustrate applications of the postulates. If \( (abc) \cdot (acd) \), then \( (a, b, d) = (a, (a, b, c), d) = ((a, a, d), b, (a, c, d)) = (a, b, c) = b \) yielding \( (abd) \). Also \( (b, c, d) = ((a, b, c), c, d) = ((a, c, d), b, (c, c, d)) = (c, b, c) = c \) so that \( (bcd) \) subsists. The converse holds by symmetry.

**Lemma 6.** In \( \mathcal{O}(a, 3) \), \( b \) is covered by \( c \neq b \): \( b <_{=c} c \iff (a, b, c) = b \) and \( (b, x, c) = b \) or \( c \) for all \( x \in 3 \).

**Proof.** Let \( b <_{=c} c \). Then \( (b, a, c) = (a, b, c) = b \) and \( (abc) \). For arbitrary \( x \in 3 \) let \( (b, x, c) = d \). Then also \( (bdc) \) by Lemma 2. Applying Lemma 5 with roles of \( c \) and \( d \) interchanged, we obtain \( (abd) \cdot (acd) \).

By Lemma 3 \( a \subseteq_b b \subseteq_d d \subseteq c \), and the hypothesis requires \( d = b \) or \( d = c \). Conversely, let \( (b, x, c) = b \) or \( c \) for all \( x \) and \( (b, a, c) = b \). Then immediately \( b \subseteq_a c \). Assume \( a \subseteq_b b \subseteq_a c \), so that \( (abx) \cdot (axc) \cdot (abc) \).

By Lemma 5 \( (bxc) \) so that \( x = (b, x, c) \), which must be \( b \) or \( c \) as desired.

**Lemma 7.** In a TDSL 3 satisfying (TF), \( bRc \) in \( g(3) \) iff \( bR_ac \) in \( g(\mathcal{O}(a, 3)) \), where \( R_a = \leq_a \). Thus \( g(3) \) and \( g(\mathcal{O}(a, 3)) \) are isometric.

**Proof.** In \( g(3) \) \( bRc \) iff \( (b, x, c) = b \) or \( c \) for all \( x \) including \( (b, a, c) = b \) or \( c \). Hence by Lemma 6 \( bRc \) iff \( b \leq_a c \) in \( \mathcal{O}(a, 3) \) iff \( bR_ac \) in \( g(\mathcal{O}(a, 3)) \).

**Lemma 8.** In a TDSL 3 satisfying (TF), (TB) and (MB) are equivalent.
Proof. Given \((abc)\). Then \((a, b, c) = b\) and \(b \subseteq \sigma a\) in the principal ideal \(\sigma(a, c)\). The latter is a finite distributive lattice by Lemma 4 and satisfies the Jordan-Dedekind chain condition. Therefore a chain 
\[a <_{a_{a_{1}}} < a \cdots < a_{m} = b \in \sigma(a, c)\]
exists and minimizes a sequence 
\[aR_{a}x_{a}R_{a} \cdots R_{a}b, \text{ where } R_{a} \subseteq \subseteq a\]. The corresponding sequence of \(g(3)\) of Lemma 7: 
\[aRa_{1}R \cdots R_{a}a_{m} = b\] is thus minimal so that \(ab = \delta_{a}[b]\), the dimension of \(b\) in \(\sigma(a, 3)\). Similarly \(ac = \delta_{a}[c]\). Again, \(b <_{a} b_{1} < \cdots <_{a} b_{m} = c\) minimizes sequences \(bRa_{1}R \cdots R_{a}c\) and by virtue of Lemma 7 yields a corresponding minimal sequence 
\[bRb_{1}R \cdots R_{b}a_{m} = c\] of \(g(3)\) of length \(bc = n\). The total chain 
\[a <_{a_{a_{1}}} < a \cdots <_{a} a_{m} = b <_{a} b_{1} < \cdots <_{a} b_{m} = c\], again in view of the Jordan-Dedekind chain condition in \(\sigma(a, c)\), yields a minimal chain 
\[aRa_{1}R \cdots R_{a}a_{m} = b\]. Hence \(ac = \delta_{a}[c] = \delta_{a}[b] + bc = ab + bc\), yielding \(abc\). Conversely suppose \(abc\) holds. Let \(d = (a, b, c)\). By Lemma 2 
\[(adb) \cdot (bdc) \cdot (cda)\]. By the proof just completed \(adb-bdc-cda\). Hence \(0 = (ad + db - ab) / 2 + (bd + dc - bc) / 2 - (cd + da - ca) / 2\) 
\[= bd - (ab + bc - ca) / 2 = bd - (ac - ca) / 2 = bd\]. Thus \(b = d\), \((a, b, c) = b\), and \((abc)\).

Proof of Theorem 2. Lemma 8 completes the hypothesis of Theorem 1.

3. Unique ternary distance graphs. We prove Theorems 3, 4 and 5 in this section.

Lemma 9. In a metric space satisfying \((U)\), \((VB)\) and \((MB)\) are equivalent.

Proof. \([bcd]\) iff \([c, a, d]\) iff \([c, b, d]\) = \(bc + cc + cd\) 
\[= (bc + cd + db) / 2\] iff \(bc + cd = bd\) iff \(bcd\).

Lemma 10. (Condition \((D)\) of Sholander [4, p. 804]). For each unordered triple \(b, c, d \in \mathfrak{F}\), a metric space satisfying \((U)\), there exists a unique \(s \in \mathfrak{F}\) such that \(bs \cdot csd \cdot dsb\), namely \(s = [b, c, d]\).

Proof. By \((U)\) there exists unique \(s = [b, c, d]\) such that \([s, b, c, d]\) = \((bc + cd + db) / 2\). We apply Lemma 1. By \((MT2)\) \(bsc \cdot csa \cdot dsb\), and for \(x \neq s\) \([x, b, c, d] > (bc + cd + db) / 2\) so that at least one of \(bxc, cxd, dxb\) fails.

Lemma 11. In any metric space \(abc \cdot acd \rightarrow abd \cdot bcd\).

This is an elementary property of metric spaces.

Proof of Theorem 3. The metric betweenness relation \(bcd\) satisfies the set of conditions \(\Sigma_{1}(D, B_{1}, F)\) of Sholander [4, pp. 803–805]: \((D)\) by Lemma 10; \((B_{1})\) \(aba \rightarrow a = b\), trivially; and \((F)\) \(abc \cdot acd\)


\[ \rightarrow \text{dba} \leftarrow \text{ab} \] by Lemma 11. By Lemma 9 the equivalent between-


ness relation \([bcd]\) also satisfies \(\Sigma_1\). Sholander showed in \([4, 4.10]\) that the corresponding ternary operation \([\text{b, c, d}]\) satisfies his condi-
tions (M) and (N). The latter, he showed in \([3, 8.3]\), are equivalent
to \((T1-2-3)\).

**Corollary to Theorem 3.** If a metric space \(\mathbb{M}\) satisfies \((U)\) and
for some pair \(\text{a, a}' \subseteq \mathbb{M}\) \(\text{axa}'\) for all \(x \in \mathbb{M}\), then \(\varphi(\text{a, a}') = \varphi(\text{a, a}')\) is a
distributive lattice with \(\text{a}\) and \(\text{a}'\) as zero and unit elements.

**Proof.** By Theorem 3 \(\mathbb{M}\) becomes a TDSL \(\mathbb{J}\) under \([\text{b, c, d}]\) with
\([\text{axa}']\) for all \(x \in \mathbb{J}\). I.e., \(\text{a} \subseteq x \subseteq \text{a}'\) for all \(x\). Lemma 4 completes the
proof.

**Lemma 12.** A necessary and sufficient condition that a connected graph
\(\mathcal{G}\) be even is that \(\text{bRc}\) implies \(\text{bx} - \text{cx} = \pm 1\). Furthermore, a UTD graph
is even.

**Proof.** Given \(\mathcal{G}\) is even and suppose \(\text{bRc}\). Then \(1 = \text{bc} \geq \text{bx} - \text{cx} \geq -\text{bc} = -1\). But \(\text{bx} \neq \text{cx}\) since \(\text{bx} - \text{cx} + 1 = \text{bx} + \text{cx} + \text{bc} \equiv 0 \pmod{2}\). Hence \(\text{bx} - \text{cx} = \pm 1\). Conversely suppose \(\mathcal{G}\) is not even. Two adjacent
vertices \(b, c\) and the opposite vertex \(x\) of a smallest odd-sided polygon
give \(\text{bc} = 1\) and \(\text{bx} = \text{cx}\). Moreover \([c; x, b, c] = \text{cx} + 1 = \text{bx} + 1 = [b; x, b, c]\)
\(\leq y + b y + c y = [y; x, b, c]\) for \(b \neq y \neq c\). Hence \(b\) and \(c\) (and possibly
\(y\)) are tied to minimal ternary distance from \(x, b, c\) so that \(\mathcal{G}\) is
not a UTD graph. We have thus proved the contrapositives of the
converse and the second statement.

We may note at this point that in a UTD graph the ternary opera-
tion \([\text{b, c, d}]\) satisfies \((T2)\) trivially by symmetry. It also satisfies
\((T1)\). For if \(x \neq a, [x; a, a, b] = ax + (ax + bx) > aa + aa + ab = [a; a, a, b]\)
so that \(a = [a, a, b]\). We shall circumvent a direct proof of \((T3)\), which
would be tedious.

**Lemma 13.** In a UTD graph \((MB)\) and \((VB)\) are equivalent.

**Proof.** First suppose \(abc\). If \(x \neq b, [x; a, b, c] = (ax + cx) + bx > ac = ab + bc = [b; a, b, c]\). Hence \(b = [a, b, c]\) and \([abc]\) subsists. Con-
versely, suppose \([abc]\). We shall prove by induction on \(n = \text{bc}\) that
\(abc\) follows. We note that \(abc\) holds trivially for \(n = 0\). When \(n = 1, \text{ab} = \text{ac} = 1\) by Lemma 12. But \(\text{ab} = \text{ac} + 1 = \text{ac} + \text{bc}\) yields \(\text{abc}\) and \([abc]\)
by the first part of this proof and leads to the contradiction \([a, b, c]\) = \([a, b, c]\) \(\neq b = [a, b, c]\). Thus \(ac = ab + bc\) as desired. Assume
\([a, b, c] = b\) implies \(abc\) whenever \(n \leq k\). Consider \([a, b, c] = b\) with
\(n = \text{bc} = k + 1\). Let \(bRb_0\) with \(b_0\) on minimal \(b-c\) chain: \(bb_0 = 1\) and
\(bb_0 + b_0c = bc\). Since \(b = [a, b, c], \text{ab} + bc = [b; a, b, c] < [b_0; a, b, c]\)
= ab_0 + (bb_0 + b_0c) = ab_0 + bc. Thus \( ab < ab_0 \) and \( ab_0 = ab + 1 \) by Lemma 12. Now \([b; a, b_0, c] = ab_0 + bc > ab + bc = ab_0 + b_0c = [b_0; a, b_0, c] \).

Also for \( x \neq b \) we apply hypothesis and Lemma 12 to obtain \([x; a, b_0, c] = [x; a, b, c] + (b_0x - bx) \geq 1 + [b; a, b, c] \geq [b; a, b_0, c] \). Uniqueness of minimality in (V) requires that \( b_0 = [a, b_0, c] \) or \([b_0c] \). But \( b_0c = bc - 1 = k \).

By the induction hypothesis \( ab_0c \) subsists. Hence \( ac = ab_0 + b_0c = ab + bc \) yielding \( abc \). The induction is complete.

**Proof of Theorem 4.** Let \( s = [b, c, d] \). Then for \( x \neq s[s; b, s, c] = (bs + cs + ds) - ds < bx + cx + (dx - ds) \leq bx + cx + sx = [x; b, s, c] \).

Thus \( s = [b, s, c] \) and \([b; c, s, b] \) subsists. Similarly \([csd] \) and \([dsb] \). Then \( bsc - csd - dsb \) by Lemma 13. By Lemma 1 and (V) we have \([s; b, c, d] = (bc + cd + db) / 2 < [x; b, c, d] \) for all \( x \neq b \). This is (U). Hence Theorem 4 now follows from Theorem 3.

**Lemma 14.** In any lattice or semi-lattice if \( aRbRcRdRa \), where \( R \) is \( \leq \), alternate \( R \)'s are opposite directional covering.

This follows by definition of covering and uniqueness of join and meet when they exist.

**Lemma 15.** In a lattice \( \mathfrak{L} \), for which \( g(\mathfrak{L}) \) is a UTD graph with respect to the ternary operation determined by the metric, \( b \leq c \) in \( \mathfrak{L} \) iff \( b \neq c \) and \([b, x, c] = b \) or \( c \) for all \( x \in g(\mathfrak{L}) \).

**Proof.** If \( b \leq c \) in \( \mathfrak{L} \), then by Lemma 12 \( g(\mathfrak{L}) \) is even and \( bx - cx = \pm 1 = \pm bc \) in \( g(\mathfrak{L}) \). Thus \([bcx] \) or \([cbx] \), i.e., \([b, x, c] = c \) or \( b \). If \( b \notin c \), then \( b = c \) or there exists \( r \in g(\mathfrak{L}) \) with \( b, r, c \) all distinct such that \([brc] \) or \([br, r, c] = r \neq b \) or \( c \).

**Proof of Theorem 5.** Finite dimensionality of the elements of \( \mathfrak{L} \) makes \( g(\mathfrak{L}) \) well defined and connected through \( z \), so that it is a UTD graph. Hence by Theorems 3 and 4 \( g(\mathfrak{L}) \) is a TDSL with respect to the operation \([a, b, c] \), and (MB) is equivalent to (VB) (which is (TB)). Moreover all the lemmas are valid and applicable. By Lemma 3 \( \varrho(z, g(\mathfrak{L})) \) is a distributive semi-lattice with the same zero element \( z \) of \( \mathfrak{L} \). We shall show that \( \varrho(z, g(\mathfrak{L})) \) is isomorphic to \( \mathfrak{L} \) under the identity correspondence \( c \leftrightarrow c \). Combining the results of Lemmas 6 and 15 \( b \leq c \in \varrho(z, g(\mathfrak{L})) \) iff \( b \leq c \) in \( \mathfrak{L} \). Accordingly, it will be sufficient to show that

(S) \( b \leq c \) in \( \varrho(z, g(\mathfrak{L})) \) implies \( b \leq c \) in \( \mathfrak{L} \).

We employ an induction on \( n = 2c \), the distance from \( z \) to \( c \) in \( g(\mathfrak{L}) \), and note (S) is trivially true for \( n = 1: z \leq c \) in \( \varrho(z, g(\mathfrak{L})) \) iff \( z \leq c \) in \( \mathfrak{L} \).

Assume (S) holds for \( n \leq k \). Now consider \( b \leq c \) with \( zb = k \) and \( zc = k + 1 \) in \( g(\mathfrak{L}) \), and assume that \( b > c \) in \( \mathfrak{L} \). From all necessarily
finite descending chains \( b > c > \cdots \) in \( \mathcal{L} \) select one with an earliest agreement of \( > \) and \( >_z \): 
\[
b = c_0 <_z c = c_1 <_z \cdots <_z c_{r-1} <_z c_r >_z c_{r+1} \leq_z \cdots
\]
By Lemma 4 the ideal \( \phi(z, c_r) \) of \( \phi(z, g(\mathcal{L})) \) is a distributive lattice. Hence by lower semi-modularity \( c_{r-1} > d = c_{r-1} \cap c_{r+1} <_z c_{r+1} \).
On the other hand \( c_{r-1} > c_r > c_{r+1} < d < c_r \), where the direction of the last two coverings are required by Lemma 14. If \( r = 1 \), the induction hypothesis requires \( b > d < c_2 \) contradicting \( c_2 < d < b \). If \( r > 1 \), then \( c_{r-1} > d \) contradicts the minimality of \( r \). Hence our assumption \( b > c \) is false, and the induction on \( n = zc \) for validity of (S) is complete.

Therefore \( \phi(z, g(\mathcal{L})) \) is isomorphic to \( \mathcal{L} \) and is a distributive lattice (rather than a semi-lattice). Thus \( \mathcal{L} \) itself is distributive.

References


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